

Operator theory

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PART I

Compact and Fredholm operators

Preliminaries

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Def: (X, ρ) - metric space if X -set, and ρ is a metric:

i) $\rho(x, y) \geq 0 \quad \forall x, y \in X. \quad \rho(x, y) = 0 \Leftrightarrow x = y$

ii) $\rho(x, y) = \rho(y, x) \quad \forall x, y \in X$

iii) $\rho(x, z) \leq \rho(x, y) + \rho(y, z) \quad \forall x, y, z \in X$

Def: $U \subseteq X$ is open if $\forall x \in U. \exists \delta > 0.$

s.t. $B(x, \delta) \subset U \quad (B(x, \delta) = \{y \in X \mid \rho(x, y) < \delta\})$

Def: $K \subset X$ is compact if every open cover $\{U_\alpha\}_{\alpha \in I}$ of K has a finite subcover.

cover: $\{U_\alpha\}_{\alpha}$ is a cover of K if $\bigcup_{\alpha \in I} U_\alpha \supset K$

Def: A precompact set $A \subset X$ is a set $A \subset X$

s.t. $\overline{A}$ is compact.
closure of A in X

Def: $\{x_j\}_{j=1}^\infty$ is **Cauchy sequence** in X if
 $\forall \varepsilon > 0. \exists N = N(\varepsilon). \rho(x_j, x_k) < \varepsilon \forall j, k \geq N(\varepsilon).$

Def: X is **complete** if $\forall$ Cauchy $\{x_j\}_{j=1}^\infty \subset X$.
 $\exists x \in X. \rho(x_j, x) \rightarrow 0$ as $j \rightarrow \infty$
 (Every Cauchy sequence converges.)

Ex. $(\mathbb{R}^n, \rho_{\mathbb{R}^n}(\{x_i\}, \{y_j\}) := \sqrt{\sum |x_i - y_i|^2})$ - complete metric space

Ex. $(\mathbb{Q}, \rho_{\mathbb{R}^n})$ - metric space but non-complete

Ex. $[0, 1]$ is a compact-subset of $(\mathbb{R}, \rho_{\mathbb{R}})$

Th: $K \subseteq \mathbb{R}^n$ is compact $\Leftrightarrow$ closed and bounded

Def: $A \subset (X, \rho)$ is bounded if $\exists x \in X. \exists R > 0. A \subset B(x, R)$

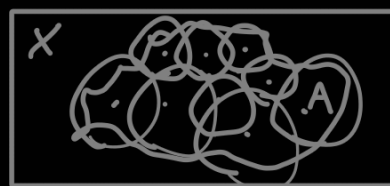
Ex: $(X, \rho) = \ell^2(\mathbb{Z}) = (\{\{x_j\}_{j \in \mathbb{Z}} \mid \sum_{j \in \mathbb{Z}} |x_j|^2 < \infty, \rho(\{x_j\}, \{y_j\}) = \sqrt{\sum_{j \in \mathbb{Z}} |x_j - y_j|^2})$

$$B[0, 1] = \{y \in \ell^2(\mathbb{Z}) : \rho(0, y) \leq 1\}$$

-bounded, closed but not compact HW

Theorem: Let (X, ρ) be a complete metric space,
 $A \subset X$. The following assertions are equivalent:

- A is precompact
- $\forall \varepsilon > 0. \exists_n$ a finite ε -net $\{x_j\}_{j=1}^{N_\varepsilon}$ in A ,
 that is, $\bigcup_{j=1}^{N_\varepsilon} B(x_j, \varepsilon) \supset A$.



iii) $\forall \{x_j\}_{j=1}^\infty \subset A$ there is a converging subsequence to some element $x \in X$.

Proof: i) $\Rightarrow$ ii) $\{\mathcal{U}_x\}_{x \in A} = \{B(x, \varepsilon)\}_{x \in A}$ - open cover of A .

- is an open cover of $\bar{A}$:

$$(\forall y \in \bar{A}. \exists x \in A. d(x, y) < \varepsilon/2 \Rightarrow y \in B(x, \varepsilon))$$

definition of closure

$\Rightarrow \exists \{U_{x_j}\}_{j=1}^N$ - finite subcover of $\bar{A}$

$\Rightarrow \{U_{x_j}\}_j$ is a ε -net in A

ii) $\Rightarrow$ iii) Observe that $\forall \varepsilon > 0$. any sequence $\{y_j\} \subset A$ has an infinite subsequence that is contained in some $B(x, \varepsilon)$. (we have a finite ε -net)

Assume that $\{y_k\}$ is arbitrary in A .

Canter diagonal process

$y_1, y_2, y_3, y_4, \dots$

$\downarrow$

$y_1, y_{k_2}, y_{k_3}, \dots$ where $\{y_{k_j}\} \subset B(x_2, 1/2)$

$\downarrow$

$y_1, y_{k_2}, y_{k_{j_3}}, y_{k_{j_4}}, \dots$ where $\{y_{k_{j_i}}\} \subset B(x_3, 1/3)$

$\downarrow$

$y_1, y_{k_2}, y_{k_{j_3}}, y_{k_{j_4}}, \dots$

Consider $z_1 = y_1$

$$z_2 = y_{k_2}$$

$$z_3 = y_{k_{j_3}}$$

$$z_4 = y_{k_{j_4}}$$

$\vdots$

Claim: $\{z_j\}$ is a Cauchy sequence.

Indeed $\rho(z_j, z_k) < \frac{1}{k}$
 $k < j$

because $z_j, z_k \in B(x_k, \frac{1}{k})$ $\rho(z_j, z_k) < \frac{2}{k} \Rightarrow \rho_k \rightarrow \infty$

X is complete $\Rightarrow \{z_j\}$ converges

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iii) $\Rightarrow$ i):

Plan: a) A contains a dense countable subset

a)+b)+c) b) If $\{U_\alpha\}_{\alpha \in I}$ is an open cover of $\bar{A}$
under assumption $\Rightarrow \exists \{U_{\alpha_j}\}_{j \in J}$ - an open countable subcover of $\bar{A}$
3) c) $\Rightarrow \{U_{\alpha_j}\}_{j=1}^\infty$ is a cover of $\bar{A}$

a) Observe that $\forall \varepsilon > 0$ there exists at most $N(\varepsilon)$ points
 $A \ni \{y_j(\varepsilon)\}_{1 \leq j \leq N(\varepsilon)}$ s.t. $\rho(y_k(\varepsilon), y_j(\varepsilon)) > \varepsilon \quad \forall k \neq j$.

(If this is not true, then $\exists \{y_j(\varepsilon)\}_{j=1}^\infty$ such that
 $\rho(y_j(\varepsilon), y_k(\varepsilon)) > \varepsilon$ and it cannot contain a convergent subsequence by Cauchy criterion.)

Now $E = \{y_k(\frac{1}{n}) \mid 1 \leq k \leq N(\frac{1}{n}), n \geq 1\}$ is a dense countable subset.

(E is dense since $\forall n, \forall x \in A, \min_{1 \leq k \leq N(\frac{1}{n})} (\rho(x, y_k(\frac{1}{n}))) \leq \frac{1}{n}$ by construction)

b) Assume that $\{U_\alpha\}_{\alpha \in I}$ is some open cover of $\bar{A}$.
For every $x \in \bar{A}$ define

$$\varepsilon(x) := \sup \{ \varepsilon > 0 \mid B(x, \varepsilon) \subset U_\alpha \text{ for some } \alpha \} > 0$$

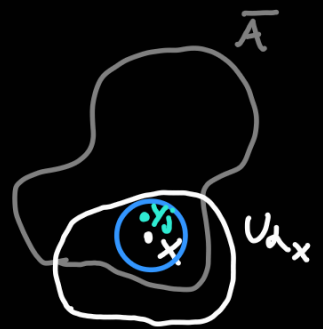
Claim: if $\{y_j\}_{j=1}^\infty$ is a countable dense subset in A , then

$\{B(y_j, \varepsilon(y_j))\}_{j=1}^{\infty}$ is an open cover for $\bar{A}$.

Take $x \in \bar{A}$ $\exists d_x$. $x \in U_{d_x}$, and let $h(x) > 0$ such that $B(x, h(x)) \subset U_{d_x}$ (U_{d_x} is open $\Rightarrow h(x) \exists$)

Find y_j such that $\rho(x, y_j) < \frac{h(x)}{10}$.

Then, since $h(x) \leq 2\varepsilon(y_j)$, $x \in B(y_j, \varepsilon(y_j)) \Leftrightarrow \rho(x, y_j) < \varepsilon(y_j)$



$\varepsilon(y_j) \geq \frac{h(x)}{10}$ - Because $B(y_j, \frac{h(x)}{5}) \subset U_{d_x}$ by triangle inequality and $\varepsilon(y_j)$ satisfies $\textcircled{*}$.

Since $\{B(y_j, \varepsilon(y_j))\}_{j=1}^{\infty}$ is an open cover for $\bar{A}$, then $\{U_{d_{y_j}}\}_{j=1}^{\infty}$ is an open cover for $\bar{A}$, where $U_{d_{y_j}}$ is the set U_{d_x} from the definition of $\varepsilon(y_j)$ (that is, $U_{d_{y_j}} \supset B(y_j, \varepsilon(y_j))$).

Since we have $\textcircled{*}$, $\bigcup_{j=1}^{\infty} U_{d_{y_j}} \supset \bar{A}$.

c) Claim: $\exists N$. $\{U_{d_{y_j}}\}_{j=1}^N$ is a cover of $\bar{A}$.

Suppose this is not the case $\Rightarrow \forall j \geq 1$. $\exists x_j \in A_j \setminus \bigcup_{k=1}^j U_{d_{y_k}}$. Consider $\{x_j\}_{j=1}^{\infty}$, and assume that the sequence $\{x_{j_k}\}$ - convergent to some $x \in X$. Note that $x \in \bar{A}$ ($x_j \in \bar{A}$). $\Rightarrow \exists j_*$. $x \in U_{d_{y_{j_*}}} \Rightarrow \exists \delta > 0$. s.t. $B(x, \delta) \subset U_{d_{y_{j_*}}}$, but $x_{j_k} \notin U_{d_{y_{j_*}}}$ for large k by construction. (in particular, $x_{j_k} \notin B(x, \delta)$, hence $\rho(x, x_{j_k}) > \delta$, but this contradicts the fact that $x_{j_k} \rightarrow x$). We have shown that $\{x_j\} \subset \bar{A}$ cannot have a convergent subsequence.

Then if $\tilde{x}_j \in A$. $\rho(\tilde{x}_j, x_j) < \frac{1}{j}$, then $\{\tilde{x}_j\}$ also has no convergent subsequence. So, we assumed there is no

finite subcover $\{U_{x_j}\}$ and found a sequence $\{\tilde{x}_j\} \subset A$ that has no converging subsequence, a contradiction with 3). Therefore $3) \Rightarrow 1)$. $\square$

Examples of compact sets and their properties:

1) $K \subset (X, \rho)$ is compact $\Rightarrow K$ is bounded

Indeed, if K is not bounded, then $\{B(x, n)\}_{n \geq 1}$ is an open cover without a finite subcover.

2) $K \subset (X, \rho)$ is compact, then it is closed
 $(\Leftrightarrow \{x_j\} \subset K$ such that $x_j \rightarrow x$ in (X, ρ) we also have $x \in K)$

Let's check that $X \setminus K$ is open. Take $y \in X \setminus K$, take $x \in K$, let $\delta(x) > 0$. $B(x, \delta(x)) \cap B(y, \delta(x)) \neq \emptyset$
 $\{B(x, \delta(x))\}_{x \in K}$ is an open cover, let $\{B(x_j, \delta(x_j))\}_{j=1}^N$ be a finite subcover, then $\delta := \min_{1 \leq j \leq N} \delta(x_j)$, $B(y, \delta) \cap K = \emptyset$
 $\Rightarrow X \setminus K$ is open.

Another proof: Suppose $\{y_j\} \subset K$ s.t. $y_j \rightarrow y$, $y \notin K$.

$$U_j = \{x \in X \mid \rho(x, y) > \frac{1}{j}\}$$

$\{U_j\}_{j=1}^{\infty}$ open cover, $\{U_{j_k}\}_{k=1}^N$ finite subcover

$$\varepsilon := \min_{1 \leq k \leq N} \left(\frac{1}{j_k} \right), \quad \rho(x, y) > \varepsilon \quad \forall x \in K \quad \text{contradiction} \quad \square$$

linear space = vector space

3) Let X be a finite-dimensional complete linear normed space. Then $E \subset X$ is compact $\Leftrightarrow E$ is closed and bounded.

($\Rightarrow$): Examples 1+2.

($\Leftarrow$): $X = \{ \sum_{k=1}^N a_k e_k \mid a_k \in \mathbb{C} \}$, $N = \dim X$, $\{e_k\}_{k=1}^N$ - basis

$$\varphi\left(\sum a_k e_k\right) = \max_{1 \leq k \leq N} |a_k| \text{ - norm on } X$$

Since all norms on finite dimensional vcc. spaces are equivalent.

$$\exists A, B > 0. \quad A\|x\| \leq \varphi(x) \leq B\|x\| \quad \forall x \in X.$$

in particular the set $\left\{ \|\varphi(x)\|_N^N, \begin{array}{l} x \in E \\ x = \sum a_k(x) e_k \end{array} \right\}$

is bounded (=bdd) in $\mathbb{C}^n$ for every bounded $E \subset X$.

$$\sup_{x \in E} \varphi(x) \leq B \cdot \sup_{x \in E} \|x\| < \infty$$

$\Rightarrow$ for any sequence $\left\{ \{c_k(x_j)\}_{k=1}^N \right\}_{j=1}^\infty$ one can extract a converging subsequence in $\mathbb{C}^n$, i.e.

$$C_k(x_{j_n}) \longrightarrow c_k \quad n \rightarrow \infty.$$

But then $\sum_{k=1}^N c_k(x_{j_n}) e_k \longrightarrow \sum c_k e_k$ in X

$\Rightarrow$ bounded subsets are precompact in X

$\Rightarrow$ bdd + closed sets are compact

$$4) \ell^2(\mathbb{Z}) = \left\{ \{c_k\}_{k \in \mathbb{Z}} \mid \sum |c_k|^2 < \infty \right\}$$

$$\|\{c_k\}\| = \sqrt{\sum_{k=1}^\infty |c_k|^2}$$

$$B[0, 1] = \left\{ \{c_k\}_{k \in \mathbb{Z}} \in \ell^2(\mathbb{Z}) \mid \|\{c_k\}\| < 1 \right\}$$

Then this set is bounded, closed, but neither compact nor precompact.

Proof: there is no finite $\frac{1}{2}$ -net in $B[0,1]$,
 beca $\rho(e_k, e_j) > \frac{1}{2}$ for $e_k = (0, \dots, 0, \overset{\text{kth place}}{1}, 0, \dots, 0)$.

Definition: X is a **Banach space** if it is a linear normed space such that X is complete with respect to this norm.

Example: Let (K, ρ) be a metric compact space.

$$C(K) := \{f: K \rightarrow \mathbb{C} \mid \text{cont. in the metric } \rho\}$$

$$(\Leftrightarrow) f(x_j) \rightarrow f(x) \quad \forall x_j \rightarrow x \text{ in } (K, \rho)$$

$$\|f\|_{C(K)} = \|f\| := \max_{x \in K} |f(x)| \quad - \text{norm in } C(K)$$

Theorem: [Arzela-Ascoli]: Assume that K is a complete compact metric space. $E \subset C(K)$ is precompact $\Leftrightarrow$

$$\Leftrightarrow \begin{cases} 1) E \text{ is bounded in } C \\ 2) \text{ Functions in } E \text{ are equicontinuous, that is,} \\ \forall \varepsilon > 0, \exists \delta_\varepsilon > 0. |f(x) - f(y)| < \varepsilon \quad \forall x, y \in K. \rho(x, y) < \delta_\varepsilon \\ \forall f \in E \end{cases}$$

We will need 1+2 $\Rightarrow$ precompactness.

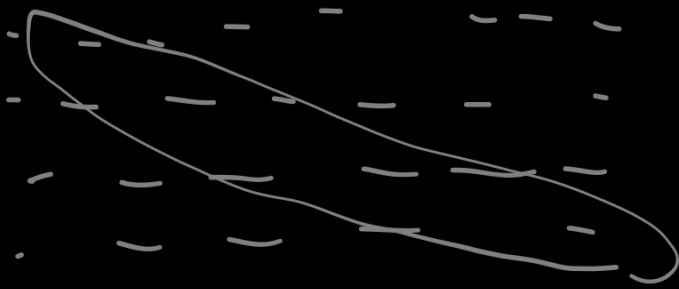
Proof: Find a dense sequence $\{x_j\}$ in K .
 (such sequence exists because K is compact)

Then take $\{f_n\}$ arbitrary sequence in E .

We want to find a converging subsequence of $\{f_n\}$
 (then E - precompact)

For this find a subsequence $\{f_{n_k}\}$ such that
 $f_{n_k}(x_j) \rightarrow f(x_j)$ for every j

(Cantor diagonalization process + uniform boundedness)



look at the proof
from the 1st
lecture

Claim: f_{n_k} is Cauchy sequence in $C(K)$.

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Aim: $\|f_{n_s} - f_{n_m}\|_{C(K)} \rightarrow 0$ as $s, m \rightarrow \infty$

For simplicity let $g_s = f_{n_s}$ $s \geq 1$.

Idea:

$$|g_s(x) - g_m(x)| \leq \underbrace{|g_s(x) - g_s(x_j)|}_{\substack{\leq \varepsilon/3 \\ \text{For all } s \text{ if } x_j \\ \text{is close to } x}} + \underbrace{|g_s(x_j) - g_m(x_j)|}_{\substack{\text{take } s, m \text{ large} \\ \text{enough:} \\ \leq \varepsilon/3}} + \underbrace{|g_m(x_j) - g_m(x)|}_{\substack{\leq \varepsilon/3 \\ \text{For all } s \text{ if } x_j \\ \text{is close to } x \\ (\text{eq. continuity})}}$$

To make the idea work we need to check that in this construction we can deal only with finite number of points x_j , $j=1, \dots, N(\varepsilon)$.

For this it suffices to find $N(\delta_\varepsilon)$ such that $\rho(x, x_j) < \delta_\varepsilon$ for every $x \in K$ and x_j , $j=1 \dots N_\varepsilon$.
($\{x_j\}_{j=1}^{N(\delta_\varepsilon)}$ is δ_ε -net).

So, it remains to show that if $\{x_j\}_{j=1}^\infty$ is dense then $\forall \delta_\varepsilon > 0. \exists N(\delta_\varepsilon). \{x_j\}_{j=1}^{N(\delta_\varepsilon)}$ is a δ_ε -net.

To this end, let $\{y_k\}_{k=1}^N$ is a $\delta_{\varepsilon/2}$ -net in K (K is compact)

Let $\{x_j\}_{j=1}^{N(\delta_\varepsilon)}$ be the part of $\{x_j\}$ such that

$$\text{dist}(\{x_j\}_{j=1}^{N(\delta_\varepsilon)}, y_k) \leq \delta_{\varepsilon/2} \quad \forall 1 \leq k \leq N.$$

$\Rightarrow$ then $\{x_j\}_{j=1}^{N(\delta_\varepsilon)}$ is a δ_ε -net by triangle inequality. $\square$

($\|g_s - g_m\|_{C(K)} \leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3$ for s, m large enough)

Example: $K = [0, 1]$, $E_A = \left\{ f \in C[0, 1], f(0) = 0, f \text{ is Lipschitz with constant at most } A \right\}, A > 0$

E_A is compact

i) E_A is bounded in $C[0, 1]$:

$$|f(x)| \leq |f(x) - f(0)| \leq A|x| \leq A$$

$$E_A \subset B(0, A)$$

ii) $|f(x) - f(y)| \leq A|x - y| \leq A\delta = \varepsilon$ if $\varepsilon > 0$, $\delta := \frac{\varepsilon}{A}$,
 $x, y \in [0, 1]: |x - y| \leq \delta$

i + ii + AA theorem $\Rightarrow E_A$ is precompact

iii) E_A is closed

if $f_n \rightarrow f$ in $C(K)$ then $f_n(0) \rightarrow f(0) \Rightarrow f(0) = 0$

$$|f_n(x) - f_n(y)| \leq A|x - y|$$

$\downarrow$

$$|f(x) - f(y)| \Rightarrow f \text{ is Lip}(A)$$

Example: Let $E = \left\{ \sum_{k \in \mathbb{Z}} c_k e^{2\pi i k x}, \text{ where } c_k \in \mathbb{C}: |c_k| \leq \frac{1}{k^2 + 1} \right\}$

Then E is compact as well in $C[0, 1]$.

i) bbs: $f \in E$. $\|f\| \leq \sum_{k \in \mathbb{Z}} \frac{1}{k^2 + 1}$

Details: exercise

ii) equicontinuity $F = \underbrace{\sum_{|k| \leq N}}_{\text{Lipschitz with some constant}} + \underbrace{\sum_{|k| > N}}_{\text{small if } N \text{ large}}$
 $A_N \sim \text{does not depend on } F$

Compact operators: basic properties

Definition: Let X, Y be Banach spaces, $T: X \rightarrow Y$ a linear map. T is called **bounded** if $T(B(0,1))$ is a bounded set in Y . T is called **compact** if $T(B(0,1))$ is a precompact set in Y . ($B(0,1) = \{x \in X : \|x\|_X < 1\}$) bounded linear operator

Some observations:

1) If $S \subset X$ is bdd then $T(S)$ is precompact compact for any bounded operator

2) T is compact $\Rightarrow T$ is bounded
(precompact sets are bounded)

3) with the norm $\|T\| = \sup_{x \in B(0,1)} \|Tx\|_Y$, the set of bdd linear operators becomes a linear normed space, to be denoted by $B(X,Y)$ or $B(X)$ if $X=Y$.

4) A linear map between Banach spaces X, Y is continuous if and only if it is bounded.

Hint: $\|Tx - Ty\| \leq \|T\| \cdot \|x - y\|$, so bounded operators are Lipschitz.

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Definition: **Banach algebra** is an associative algebra which is a linear space with a norm $\|\cdot\|$ such that it's a Banach space with respect to this norm (it is complete with respect to this norm) and $\|T_1 T_2\| \leq \|T_1\| \|T_2\|$ for any elements in this algebra.

Proposition: Let X be a Banach space, then $B(X,X)$ is a Banach algebra.

Proof: $\alpha_1 T_1 + \alpha_2 T_2 \in \mathcal{B}(X, X) \quad \forall T_1, T_2 \in \mathcal{B}(X, X)$ (proved)

$T_1 \cdot T_2 \in \mathcal{B}(X, X)$, since $\forall x \in X, \|T_1 T_2 x\| \leq \|T_1\| \cdot \|T_2 x\|$

$$\Rightarrow \|T_1\| \cdot \|T_2\| \leq \|T_1\| \cdot \|T_2\|$$

$$\sup_{\substack{y \in X \\ \|y\| \leq 1}} \|T_1 T_2 y\|$$

since $T_1 \in \mathcal{B}(X, X)$ we have

$$\|T_1 y\| \leq \|T_1\| \|y\| \quad \forall y \in X$$

We see that $T_1 T_2 \in \mathcal{B}(X, X)$ and $\|T_1 T_2\| \leq \|T_1\| \|T_2\|$.

Now let us prove that $\mathcal{B}(X, X)$ is Banach.

Let us show that $\sum_{n=1}^{\infty} B_k$ converges if $\sum_{k=1}^{\infty} \|B_k\| < \infty$. (*)

$$T_n = \sum_{k=1}^n B_k, \quad x \in X, \quad \|T_N x - T_M x\| = \left\| \sum_{k=M+1}^N B_k x \right\| \leq \sum_{k=M+1}^N \|B_k\| \cdot \|x\| \xrightarrow{N, M \rightarrow \infty} 0,$$

because of (*).

$\Rightarrow \{T_N x\}_N$ Cauchy in X , but X -Banach $\Rightarrow \exists T x = \lim_{N \rightarrow \infty} T_N x$

$$\text{Moreover, } \|T x\| = \lim_{N \rightarrow \infty} \|T_N x\| \leq \lim_{N \rightarrow \infty} \sum_{k=1}^N \|B_k\| \|x\| \leq \left(\sum_{k=1}^{\infty} \|B_k\| \right) \cdot \|x\|.$$

$$\Rightarrow T \in \mathcal{B}(X, X), \quad \|T\| \leq \sum_{k=1}^{\infty} \|B_k\|$$

$$\sup_{\substack{x \in X \\ \|x\| \leq 1}} \|T x - T_N x\| = \lim_{N \rightarrow \infty} \|T_N x - T_M x\| \leq \sup_{\|x\| \leq 1} \lim_{N \rightarrow \infty} \sum_{k=M+1}^N \|B_k\| \cdot \|x\| = \sum_{k=M+1}^{\infty} \|B_k\| \xrightarrow{M \rightarrow \infty} 0 \quad (*) \quad \square$$

$= S_{\infty}(X, X)$ (index ∞ will be explained later)

Proposition: The set $S_{\infty}(X)$ of all compact operators on X is a two-sided ideal in $\mathcal{B}(X) = \mathcal{B}(X, X)$: $\forall T_1 \in S_{\infty}(X), \forall T_2 \in \mathcal{B}(X), T_1 T_2 \in S_{\infty}(X)$ and $T_2 T_1 \in S_{\infty}(X)$.

Proof: Take $\{x_n\}_1^{\infty}$ s.t. $\|x_n\| \leq 1$, and let us check that there is a subsequence $\{x_{n_k}\}$: $T_1 T_2 x_{n_k}$ converges.

Note that $\{T_2 x_{n_k}\} \subset B_X(0, \|T_2\|)$. T_1 takes $B_X(0, \|T_2\|)$ into a precompact subset of $X \Rightarrow \exists \{T_1 T_2 x_{n_k}\}$ -convergent subsequence

Now let's consider $\{T_2 T_1 x_n\}$. Note that $\{T_1 x_{n_k}\}$ -convergent subsequence ($T_1 \in S_{\infty}(X)$). Then $\{T_2 T_1 x_{n_k}\}$ converges, since T_2 is continuous. ▣

Proposition: $S_\infty(X, Y)$ is a closed subset in $\mathcal{B}(X, Y)$, i.e.
 $T_n(X, Y), T_n \rightarrow T$ in $\mathcal{B}(X, Y) \Rightarrow T \in S_\infty(X, Y)$?

Proof: Let's find a finite ε -net in $T(B_X[0, 1])$.
 Take finite $\varepsilon/3$ -net for $T_n B_X(0, 1)$ for n : $\|T - T_n\| \leq \varepsilon/2$;
 denote it by $\{x_k\}_{k=1}^N$, then

$$\begin{aligned} \|Tx - Tx_k\| &\leq \underbrace{\|Tx - T_n x\|}_A + \underbrace{\|T_n x - T_n x_k\|}_B + \underbrace{\|T_n x_k - Tx_k\|}_C \\ &\leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3 \leq \varepsilon. \end{aligned}$$

✓ for any $1 \leq k \leq N$
 so choose k : $B \leq \varepsilon/2$
 and note $A \leq \|T - T_n\| \leq \varepsilon/3$
 for every $x \in B_X(0, 1)$
 $C \leq \varepsilon/3$

Corollary: If T is a limit of finite-rank operators in $\mathcal{B}(X, Y)$,
 then $T \in S_\infty(X, Y)$.

Proof: Since finite-rank operators are in $S_\infty(X, Y)$, we
 have $T \in S_\infty(X, Y)$ by the previous proposition. $\square$

Remark: At a general Banach space $\exists T \in S_\infty(X, Y)$ such
 that $\nexists \{T_n\}_n$: $\text{rank } T_n < \infty$ and $\|T - T_n\| \rightarrow 0$.

Definition: Let X be a Banach space. $\{e_k\}_{k=1}^\infty$ is a
Schauder basis if $\forall x \in X. \exists! \{c_k(x)\}_{k=1}^\infty$ such that
 $x = \sum_{k=1}^\infty c_k(x) e_k$, where the series converges in X .

Theorem: Let X be a Banach space with Schauder
 basis, then $T \in S_\infty(X) \Leftrightarrow \exists T_n. \text{rank } T_n \leq n$ and $\|T - T_n\| \rightarrow 0$.
 (here $\text{rank } S = \dim S(X) \quad \forall S \in \mathcal{B}(X)$)

Proof: ($\Leftarrow$): we already know

($\Rightarrow$): Let $T \in S_\infty(X)$, and let $P_n: X \rightarrow \sum_{k=1}^n C_k(x) e_k$.

P is linear: $\forall \alpha, \beta \in \mathbb{C}, \forall x, y \in X$. $P_n(\alpha x + \beta y) = \alpha P_n(x) + \beta P_n(y)$?

$$\left. \begin{array}{l} \text{If } x = \sum C_k(x) e_k \\ y = \sum C_k(y) e_k \end{array} \right\} \Rightarrow \alpha x + \beta y = \sum_{k=1}^{\infty} (\alpha C_k(x) + \beta C_k(y)) e_k$$

$$\alpha x + \beta y = \sum_{k=1}^{\infty} C_k(\alpha x + \beta y) e_k$$

by uniqueness by def. of Schauder basis

$$C_k(\alpha x + \beta y) = \alpha C_k(x) + \beta C_k(y) \quad \forall k$$

$$\begin{aligned} \text{Then } P_n(\alpha x + \beta y) &= \sum_{k=1}^n C_k(\alpha x + \beta y) e_k = \sum_{k=1}^n \alpha C_k(x) e_k + \sum_{k=1}^n \beta C_k(y) e_k \\ &= \alpha P_n(x) + \beta P_n(y) \Rightarrow P_n \text{ linear} \end{aligned}$$

Note that $T_n := P_n T$ are such that $\text{rank}(T_n) \leq n$ because $\dim P_n T(x) \leq \dim P_n(x) \leq n$.

It remains to show that $T_n \rightarrow T$ in $\mathcal{B}(X)$. Since T is compact, $\forall \varepsilon > 0, \exists \{x_k\}_{k=1}^N$ such that $\|x_k\| \leq 1 \quad \forall k$ and $\{Tx_k\}_{k=1}^N$ is a ε -net in $T(B_X(0,1))$. Now take $x \in B_X(0,1)$ and write

$$\begin{aligned} \|Tx - T_n x\| &\leq \|Tx - Tx_k\| + \|Tx_k - T_n x\| + \|T_n x_k - T_n x\| \\ &\leq \underbrace{\|Tx - Tx_k\|}_{\leq \varepsilon \text{ for some } k} + \underbrace{\|Tx_k - P_n Tx_k\|}_{\leq \varepsilon \text{ if } n \text{ large enough for any fixed } k} + \underbrace{\|P_n Tx_k - P_n Tx\|}_{\leq \|P_n\| \cdot \|Tx_k - Tx\|} \\ &\leq \varepsilon + \varepsilon + \underbrace{\sup_{k \leq N} \|P_n\|}_{\leq \infty} \varepsilon \end{aligned}$$

$\sup \|P_n\| < \infty$ by Banach-Schteinhaus theorem theorem on uniform point-wise convergence.

$$\|T - T_n\| \leq \varepsilon (2 + \sup \|P_n\|) \quad \text{for } n \text{ large enough}$$



Theorem [Banach-Schteinhaus]: Assume that $\{T_n\}_{n=1}^{\infty} \subset \mathcal{B}(X)$ where X is a Banach space, such that

$$\sup_n \|T_n x\| \leq C(x) < \infty \quad \begin{array}{l} \text{local information} \\ \sim \text{uniform estimate} \end{array}$$

for every $x \in X$. Then $\sup_n \|T_n\| < \infty$. In particular, one can take C in place of $\sup_n C(x)$.

Remark: In our situation $\sup_{1 \leq n \leq \infty} \|P_n\| < C(x) < \infty$ because $P_n x \rightarrow x$ in X .

Banach adjoint Operators

October 15, 2025

Definition: Let X be a Banach space, then $X^* = \mathcal{B}(X, \mathbb{C})$ is called the dual space to X . The elements of X^* are called functionals.
 $\{\phi: X \rightarrow \mathbb{C} \mid \begin{array}{l} \phi \text{ linear} \\ \|\phi\| < \infty \end{array}\}$

Examples: (can ignore, if one does not know measure theory)

$$i) L^p(\mu) = \left\{ f: S \rightarrow \mathbb{C} \mid \begin{array}{l} f \text{ is measurable with} \\ \text{respect to } \sigma\text{-algebra} \\ \text{of } \mu \end{array}, \int_S |f|^p d\mu < \infty \right\}$$

$f=g$
(if $|f| \leq g(x)$
for μ -a.e. $x \in S$)

$$\|f\|_{L^p(\mu)} = \left(\int_X |f|^p d\mu \right)^{1/p}$$

$$(L^p(\mu))^* = L^q(\mu) \quad \text{where} \quad \frac{1}{p} + \frac{1}{q} = 1$$

$$ii) \ell^p(\mathbb{Z}) = \left\{ \{x_k\}_{k \in \mathbb{Z}} \mid \sum_{k \in \mathbb{Z}} |x_k|^p < \infty \right\}, \quad \|x\|_{\ell^p(\mathbb{Z})} = \left(\sum |x_k|^p \right)^{1/p}$$

$$\ell^p(\mathbb{Z})^* = \ell^q(\mathbb{Z}), \quad \text{where} \quad \frac{1}{p} + \frac{1}{q} = 1$$

In these examples, the following identification is assumed:

$$i) g \in L^q(\mu) \leftrightarrow \phi_g: f \mapsto \int f g d\mu, \quad \phi_g: L^p(\mu) \rightarrow \mathbb{C}$$

$$ii) \{x_k\}_{k=1}^\infty \text{ in } \ell^q(\mathbb{Z}) \leftrightarrow \phi_{\{x_k\}}: \{x_k\} \mapsto \sum_{k \in \mathbb{Z}} x_k y_k$$

$$\phi_{\{x_k\}}: \ell^p(\mathbb{Z}) \rightarrow \mathbb{C}$$

Remark: i) is non-trivial measure theory

More examples:

$$iii) C_0(\mathbb{Z}) = \left\{ \{x_k\}_{k \in \mathbb{Z}} \mid x_k \rightarrow 0 \text{ as } |k| \rightarrow \infty \right\}$$

$$C_0^*(\mathbb{Z}) = \ell^1(\mathbb{Z})$$

same identification

(Hausdorff is actually sufficient $\leftrightarrow$ hard)

iv) Let K be a compact metric space, and $X = C(K)$.
Then $X^* = \mathcal{M}(K)$.

set of
cont. maps
 $(K, \mathcal{B}) \rightarrow \mathbb{C}$

$$\left\{ \begin{array}{l} \text{the set of Borel} \\ \text{measures on } K \\ \text{(complex valued)} \end{array} \right\} \quad \|\mu\|(K) = \sup_{\substack{K = \bigcup E_n \\ E_n \cap E_j = \emptyset \\ n \neq j}} \sum_{n \in \mathbb{Z}} |\mu(E_n)| < \infty$$

$$\|\mu\| = \|\mu\|(K)$$

Riesz - Markov representation theorem

$$\text{Here } \mu \in \mathcal{M} \xleftrightarrow{(\approx)} \phi_\mu: f \mapsto \int_K f d\mu$$

We can also define ℓ^p, L^p for $p = \infty$:

$$\cdot \quad \ell^\infty(\mathbb{Z}) := \left\{ \{x_k\} \subset \mathbb{C} : \sup_{k \in \mathbb{Z}} |x_k| < \infty \right\}$$

$$\cdot \quad L^\infty(\mu) := \left\{ f: \text{---} \text{---} : \text{ess sup } |f| < \infty \right\}$$

Remark: If $1 < p < \infty$ then $(L^p)^* = L^q$, $(L^q)^* = L^p$

Bvt for $p=1$ $(L^1)^* = L^\infty$, $(L^\infty)^* \neq L^1$
 $\ell^1(\mathbb{Z})^* = \ell^\infty(\mathbb{Z})$, bvt $(\ell^\infty(\mathbb{Z}))^* \neq \ell^1(\mathbb{Z})$

Definition: Let X, Y be Banach spaces, $T \in \mathcal{B}(X, Y)$. Then $T^* \in \mathcal{B}(Y^*, X^*)$ is defined by

$$T^*: \Psi \longmapsto \left((T^*\Psi): x \mapsto \langle Tx, \Psi \rangle \right),$$

where $\langle x, \phi \rangle = \phi(x)$ for $x \in X, \phi \in X^*$. $\Psi(Tx)$

Remark: $\langle Tx, \Psi \rangle = \langle x, T^*\Psi \rangle \rightarrow$ this formula is equivalent to the definition of T^*

Remark: Operation that sends x, ϕ into $\phi(x) = \langle x, \phi \rangle$ for $x \in X, \phi \in X^*$ is called a **pairing** of Banach spaces X, X^* .

Example: For $f \in C[0, 1], \mu$ on $[0, 1]$ then the pairing is $\langle f, \mu \rangle = \int_0^1 f d\mu$, see (*).

Theorem: Let X, Y be Banach spaces, $T \in \mathcal{B}(X, Y)$. Then the map $T^*: Y^* \rightarrow X^*$ defined by $\langle x, T^*\Psi \rangle := \langle Tx, \Psi \rangle, x \in X$, is an element of $\mathcal{B}(Y^*, X^*)$.
 $(\Leftrightarrow) (T^*\Psi)(x) = \Psi(Tx)$

Lemma 1 [Hahn-Banach theorem]: Let X be a Banach space, $E \subset X$ - subspace in X , $\phi_0: E \rightarrow \mathbb{C}$ is linear and bdd ($\Leftrightarrow \phi_0 \in E^*$). Then $\exists \phi \in X^*$ such that $\phi|_E = \phi_0$ and $\|\phi\| = \|\phi_0\|$.

Lemma 2 ["sufficient amount of functionals"]:

Let $x \in X$, then $\|x\| = \sup_{\|\phi\| \leq 1} |\phi(x)|$.

Proof: $|\phi(x)| \leq \|\phi\| \cdot \|x\| \leq \|x\|$, so $\|x\| \geq \sup_{\|\phi\| \leq 1} |\phi(x)|$

To prove " $\leq$ ", define $E = \text{span}\{x\} = \{\lambda x, \lambda \in \mathbb{C}\}$,
 $\phi_0: E \rightarrow \mathbb{C}$ if $y = c_y x \in E$.

Assume that $\|x\|=1$, then $\|\phi_0\|_{E^*} = \sup_{\|y\| \leq 1} |c_y| =$ (from $(**)$,
 $|c_y| = \|y\|$ if $\|x\|=1$) $= \sup_{\|y\| \leq 1} \|y\| = 1$.

Hahn-Banach theorem $\Rightarrow \exists \tilde{\phi}_0 \in X^*: \|\tilde{\phi}_0\|=1, \phi|_E = \phi_0$.

In particular $\sup_{\|\phi\| \leq 1} |\phi(x)| \geq |\tilde{\phi}_0| = |\phi_0(x)| = 1 = \|x\|$.

We have proved " $\leq$ " in the case where $\|x\|=1$.

The general case follows from consideration of $\frac{x}{\|x\|}$ in place of x .

October 21, 2025

We are proving that $T \in \mathcal{B}(X, Y) \Rightarrow T^* \in \mathcal{B}(Y^*, X^*)$ and $\|T\| = \|T^*\|$.

Let $T \in \mathcal{B}(X, Y)$, consider

$$\|T^*\| = \sup_{\substack{\psi \in Y^* \\ \|\psi\| \leq 1}} \|T^* \psi\|_{X^*} = \sup_{\substack{\psi \in Y^* \\ \|\psi\| \leq 1}} \sup_{\|x\| \leq 1} |(T^* \psi)(x)|$$

$$= \sup_{-1} \sup_{-1} |\langle x, T^* \psi \rangle|$$

$$= \sup \sup |\langle T x, \psi \rangle|$$

$$= \sup_{\substack{\psi \in Y^* \\ \|\psi\| \leq 1}} \sup_{\|x\| \leq 1} |\psi(Tx)|$$

$$= \sup_{\|x\| \leq 1} \left(\sup_{\substack{\psi \in Y^* \\ \|\psi\| \leq 1}} |\psi(Tx)| \right)$$

$= \|Tx\|$ Lemma "sufficient amount of functionals"

$$= \sup_{\|x\| \leq 1} \|Tx\| = \|T\| < \infty$$

The claim follows. ▣

Corollary: $T \in \mathcal{B}(X, Y)$ is invertible ($\exists T^{-1} \in \mathcal{B}(Y, X)$) iff $T^* \in \mathcal{B}(Y^*, X^*)$ is invertible ($\exists (T^*)^{-1} \in \mathcal{B}(X^*, Y^*)$).

We prove just $\Rightarrow$.

Proof: Assume that T is invertible $\Leftrightarrow T^{-1}T = I_X$
 $TT^{-1} = I_Y$

Let's take adjoint operators and see:

$$\left. \begin{aligned} (T^{-1}T)^* &= (I_X)^* \\ (TT^{-1})^* &= (I_Y)^* \end{aligned} \right\} \Leftrightarrow \left\{ \begin{aligned} T^*(T^{-1})^* &= I_X^* \\ (T^{-1})^*T^* &= I_Y^* \end{aligned} \right.$$

Exercise: $(AB)^* = B^*A^*$

It remains to check that $I_X^* = I_{X^*}$, $I_Y^* = I_{Y^*}$. Then, by the previous theorem, $(T^{-1})^* \in \mathcal{B}(X^*, Y^*)$, hence T^* is invertible and its bounded inverse is $(T^*)^{-1} = (T^{-1})^*$.

Let's check that $I_X^* = I_{X^*}$. Take $\Phi \in X^*$, $x \in X$.

$$(I_X^* \Phi)(x) = \langle x, I_X^* \Phi \rangle = \langle I_X x, \Phi \rangle = \langle x, \Phi \rangle = \Phi(x)$$

$$(I_X^* \Phi)(x) \stackrel{\text{def}}{=} (\Phi)(x) = \Phi(x).$$

Similarly, $I_Y^* = I_{Y^*}$. ▣

The „pairing notation“ is often not used in literature, but it is very useful to not make mistakes.

Theorem [Schauder]: We have $T \in S_\infty(X, Y) \Leftrightarrow T^* \in S_\infty(Y^*, X^*)$.

Proof: We will prove just " $\Rightarrow$ ".

Consider $K = \overline{TB_X(0, 1)}$ - a compact set. Let $C(K)$ be the Banach space of continuous functions on K with

$$\|f\|_{C(K)} = \max_{s \in K} |f(s)|, \quad f: K \rightarrow \mathbb{C}$$

Let $E := \{\psi \in Y^* \mid \|\psi\|_{Y^*} \leq 1, \psi \text{ is considered as a function on } K\}$

$K \subset Y$, K metric space with respect to the metric $\rho(y_1, y_2) = \|y_2 - y_1\|_Y$

So, $E \subset C(K)$ and we claim that E is precompact.

1) Uniform boundedness:

$$\psi \in E \Rightarrow \|\psi\|_{C(K)} = \max_{s \in K} |\psi(s)| \stackrel{\psi \text{ cont.}}{=} \max_{s \in \overline{TB_X(0, 1)}} |\psi(s)| = \sup_{x \in B_X(0, 1)} |\psi(Tx)|$$

$$\leq \|\psi\| \sup_{\|x\| \leq 1} \|Tx\| \leq \|\psi\| \cdot \|T\| \leq \|T\| < \infty$$

does not depend on ψ

2) Equicontinuity: take $s_1, s_2 \in K$, let's estimate

$$|\psi(s_1) - \psi(s_2)| = |\psi(s_1 - s_2)| \leq \|\psi\| \cdot \|s_1 - s_2\| \leq \|s_1 - s_2\|,$$

so maps from E are Lipschitz with constant 1, hence equicontinuous.

$\Rightarrow$ By Arzela-Ascoli theorem, E is precompact.

We are now ready to prove $T^* \in S_\infty(Y^*, X^*)$. For this, we need to check that if $\{\psi_n\}$ is a sequence in $B_{Y^*}(0, 1)$, then $\exists \{\psi_{n_k}\}$ such that $T^* \psi_{n_k}$ converges in X^* . So, take $\{\psi_n\} \subset B_{Y^*}(0, 1)$ and consider it as elements $E \subset C(K)$.

Let $\{\psi_{n_k}\}$ be such that $\psi_{n_k} \xrightarrow{\|\cdot\|} \psi$ in $C(K)$. (E is precompact!)

Let's prove that $\{T^* \psi_{n_k}\}$ is Cauchy in X^* , then the theorem will follow.

Take $x \in X, \|x\| \leq 1$, and consider

$$\begin{aligned} \|(T^* \psi_{n_k})(x) - (T^* \psi_{n_j})(x)\| &= \|\langle x, T^* \psi_{n_k} \rangle - \langle x, T^* \psi_{n_j} \rangle\| \\ &= \|\langle Tx, \psi_{n_k} \rangle - \langle Tx, \psi_{n_j} \rangle\| \\ &= \|\psi_{n_k}(Tx) - \psi_{n_j}(Tx)\| \\ &\leq \sup_{s \in K} |\psi_{n_k}(s) - \psi_{n_j}(s)| \\ &= \underbrace{\|\psi_{n_k} - \psi_{n_j}\|_{C(K)}}_{\varepsilon_{k,j} \text{ - does not depend on } x} \longrightarrow 0 \text{ by } (*). \end{aligned}$$

$$\Rightarrow \|T^* \psi_{n_k} - T^* \psi_{n_j}\| \leq \varepsilon_{k,j} \longrightarrow 0. \quad \square$$

Fredholm alternative

Example: Consider the equation $f(t) - \int_0^1 e^{t-s} f(s) ds = g(t)$ in $L^2[0,1]$. ^(**)

Question: For which $g \in L^2[0,1]$ do we have a solution $f \in L^2[0,1]$?

Observation: g has to satisfy $\int_0^1 e^{-t} g(t) dt = 0$ ^(***)

$$\text{Indeed, } \int_0^1 e^{-t} g(t) dt = \int_0^1 e^{-t} f(t) dt - \int_0^1 e^{-t} \left(\int_0^1 e^{t-s} f(s) ds \right) dt = 0 \quad \int_0^1 c dt = c$$

It is not clear so far if there are other restrictions.

Theorem [Fredholm alternative]: Let X be a Banach space, $T = I - K$ for $K \in S^\infty(X, X)$. Then

$$\text{Ran } T = \{x \in X \mid \langle x, \phi \rangle = 0 \ \forall \phi \in \ker T^*\}.$$

In other words, either:

- (1) $\ker T^* = \{0\}$ and the equation $Tf = g$ has solution $\forall g \in X$.
- (2) $\ker T^* \neq \{0\}$ and the equation $Tf = g$ has solutions only for $g : \langle g, \phi \rangle = 0 \ \forall \phi \in \ker T^*$.

Let's complete the consideration of the example:
 we need to check that $K: f \rightarrow \int_0^1 e^{t-s} f(s) ds$ is compact (exercise) and find $\text{Ker } T^*$.

$$\phi \in \text{Ker } T^* \Leftrightarrow T^* \phi = 0$$

Adjoint operator T^* is defined by

$$\begin{aligned} \langle Tf, g \rangle &= \langle f, T^*g \rangle & f, g \in L^2[0,1] \\ & & (L^2[0,1]^* = L^2[0,1]) \\ \Leftrightarrow \langle f - \int_0^1 e^{t-s} f(s) ds, g \rangle &= \int_0^1 f(t) g(t) dt - \int_0^1 \int_0^1 e^{t-s} f(s) ds g(t) dt \\ &= \int_0^1 f(t) g(t) dt - \int_0^1 f(s) \left(\int_0^1 e^{t-s} g(t) dt \right) ds \\ &= \langle f, g - \int_0^1 e^{t-s} g(t) dt \rangle_{L^2[0,1]} \end{aligned}$$

$$(T^*g): s \mapsto g(s) - \int_0^1 e^{t-s} g(t) dt, \quad s \in [0,1].$$

$$T^*g = 0 \Leftrightarrow g(s) = \int_0^1 e^{t-s} g(t) dt \quad \text{a.e. on } [0,1]$$

$$\Leftrightarrow e^s g(s) = \underbrace{\int_0^1 e^t g(t) dt}_{\text{constant}} \quad \text{for almost every } s \in [0,1]$$

$$\Rightarrow \text{So, } \text{Ker } T^* = \{ c \cdot e^{-s}, c \in \mathbb{C} \}, \quad \dim(\text{Ker } T^*) = 1.$$

By Fredholm theorem, equation (***) is solvable $\Leftrightarrow$

$$\forall c \in \mathbb{C}. \langle g, c \cdot e^{-s} \rangle = 0 \Leftrightarrow \int_0^1 g(s) e^{-s} ds = 0, \text{ which is } (***).$$

Preliminaries

Lemma [almost orthogonality in Banach spaces]: Let X be a Banach space, $E \subseteq X$ - a linear closed subspace, $\varepsilon > 0$. Then $\exists x_0 \in X$ such that $\|x_0\| = 1$, $\text{dist}(x_0, E) \geq 1 - \varepsilon$.

Proof: Since $E \neq X$, then $\exists \tilde{x}_0 \in X \setminus E$. Since E is closed, we have $\text{dist}(\tilde{x}_0, E) = \delta > 0$ for some $\delta > 0$. Now consider $\tilde{y}_0 \in E$ such that $\delta \leq \|\tilde{x}_0 - \tilde{y}_0\| \leq (1+\eta)\delta$ for some $\eta \in (0,1)$.

Now let $x_\eta := \frac{\tilde{x}_0 - \tilde{y}_0}{\|\tilde{x}_0 - \tilde{y}_0\|}$, $\|x_\eta\| = 1$.

$$\begin{aligned} \text{dist}(x_\eta, E) &= \frac{1}{\|\tilde{x}_0 - \tilde{y}_0\|} \text{dist}(\tilde{x}_0 - \tilde{y}_0, E) \\ &= \frac{1}{\|\tilde{x}_0 - \tilde{y}_0\|} \text{dist}(\tilde{x}_0, E) \\ &= \frac{\delta}{\|\tilde{x}_0 - \tilde{y}_0\|} \geq \frac{1}{1+\eta} \end{aligned}$$

Choosing η so that $\frac{1}{1+\eta} = 1 - \varepsilon$, we are done.

October 22, 2025

Lemma: Let X be a Banach space. Then $I: x \mapsto x$ is compact on $X \Leftrightarrow \dim X < \infty$.

Proof: $\dim X < \infty \Rightarrow I \in S_\infty(X)$ - we already know
 $I \in S_\infty(X) \Rightarrow \dim X < \infty$:

Suppose $\dim X = +\infty$, find a sequence $\{e_n\}$: $\|e_n\| = 1$
 $\forall n \in \mathbb{N}$

$e_1 \in X$ - arbitrary

e_2 : $\text{dist}(e_2, \text{span}\{e_1\}) \geq 1/2$

e_3 : $\text{dist}(e_3, \text{span}\{e_1, e_2\}) \geq 1/3$

e_4 : etc

existence of $\{e_n\}$
 follows from previous
 lemma, because
 $\text{span}\{e_1, \dots, e_n\} \neq X$
 $\forall n \in \mathbb{N}$

Then $\{e_n\} \subset B_X[0,1] = I(B_X[0,1])$ but there is no convergent subsequence, because $\|e_k - e_j\| \geq 1/2 \forall k, j$. $\square$

Lemma: Let X be a Banach space, $K \in S_\infty(X)$, $T = I - K$.

Then: 1) $\dim(\ker T) < \infty$.

2) $\text{Ran } T$ is closed in X . [closed range Lemma]

Proof: 1): We have $I|_{\ker T} = \underbrace{(I-K)|_{\ker T}}_0 + \underbrace{K|_{\ker T}}_{\in S_\infty(\ker T, S)}$

$$I|_{\ker T} \in S_\infty(\ker T, X) \Rightarrow I \in S^\infty(\ker T) \Rightarrow \dim(\ker T) < \infty$$

2): The statement is equivalent to the fact that if $\{x_n\} \subset X$ s.t. $Tx_n \rightarrow y$ in X then $\exists x \in X$. $Tx = y$.

2.a) Let $\{x_n\} : \|x_n\| \leq C \quad \forall n$.

Then $(I-K)(x_n) \rightarrow y$, $(I-K)(x_{n_k}) \xrightarrow{(*)} y$
For every subsequence x_{n_k}

Let's choose x_{n_k} : Kx_{n_k} converges to $z \in X$
(use $K \in S_\infty(X)$)

Then $x_{n_k} \rightarrow y+z$ by $(*)$, take $x = y+z$:

$$T(y+z) = \lim_{k \rightarrow \infty} Tx_{n_k} = y, \text{ so } Tx = y. \quad \checkmark$$

2.b) $\text{dist}(x_n, \ker T) \leq C \quad \forall n \in \mathbb{Z}$

Take $\tilde{x}_n := x_n - w_n$, where $w_n \in \ker T : \|\tilde{x}_n\| \leq 2C$.

We have $\lim_{n \rightarrow \infty} T\tilde{x}_n = \lim_{n \rightarrow \infty} Tx_n$ by step 2a) $\exists \tilde{x} : T\tilde{x} = y \quad \checkmark$

2.c) $\text{dist}(x_n, \ker T) \rightarrow +\infty$. Let us show that this situation does not occur. Suppose the converse:

Consider $\tilde{x}_n = x_n - \underbrace{w_n}_{\in \ker T} : \text{dist}(x_n, \ker T) \leq \|\tilde{x}_n\| \leq 2\text{dist}(x_n, \ker T)$
(**)

For $z_n = \frac{\tilde{x}_n}{\|\tilde{x}_n\|}$ we have $Tz_n \rightarrow 0$.

$$Tz_n = T \frac{\tilde{x}_n}{\|\tilde{x}_n\|} = T \frac{x_n}{\|x_n\|} = \frac{Tx_n}{\|x_n\|} \rightarrow \begin{matrix} \nearrow \neq \Rightarrow Tx_n \text{ is bdd in } X \\ \searrow +\infty \end{matrix}$$

$$\Rightarrow \|Tz_n\| \leq \frac{2\|y\|}{\|x_n\|} \rightarrow 0 \text{ for } n \text{ large enough}$$

At the same time, $Tz_n = z_n - Kz_n$

$\Rightarrow \exists \{z_{n_k}\}$ s.t. $\{Kz_{n_k}\}$ converges to some $z \in X$

$$\Rightarrow z_{n_k} = \underbrace{Tz_{n_k}}_{\rightarrow 0} + \underbrace{Kz_{n_k}}_{\rightarrow z} \rightarrow z$$

We have $Tz = 0$ ($= \lim Tz_{n_k} = \lim Tz_n = 0$)

$$\Leftrightarrow z \in \ker T, \quad 0 = \text{dist}(z, \ker T) =$$

$$= \lim_{k \rightarrow \infty} \text{dist}(z_{n_k}, \ker T)$$

$$= \lim_{k \rightarrow \infty} \text{dist}\left(\frac{\tilde{x}_{n_k}}{\|\tilde{x}_{n_k}\|}, \ker T\right)$$

$$= \lim \frac{\text{dist}(\tilde{x}_{n_k}, \ker T)}{\|\tilde{x}_{n_k}\|}$$

$$= \lim \frac{\text{dist}(x_{n_k}, \ker T)}{\|\tilde{x}_{n_k}\|}$$

$$\stackrel{(*)}{\geq} \frac{1}{2} \leadsto \text{contradiction} \quad \square$$

Lemma: Let X be a Banach space, $T \in \mathcal{B}(X)$:
 $\ker(T) = \{0\}$ and $T^{k+1}X = T^kX$ for some $k \geq 0$.
 Then $\text{Ran } T = X$.

Proof: We need to prove that $\forall a \in X. \exists \tilde{a} \in X. T\tilde{a} = a$.

We know that: $T^{k+1}a = T^k \tilde{a}$ for every a and some $\tilde{a}$ depending on a .
 $\checkmark \ker T^k \neq \{0\} \Rightarrow \ker T \neq \{0\}$

$$\Rightarrow T^k(a - T\tilde{a}) = 0 \Rightarrow a - T\tilde{a} = 0 \Rightarrow a = T\tilde{a}. \quad \square$$

Theorem [Fredholm]: Let X be a Banach space.

$K \in S_\infty(X)$, $T = I - K$. Then TFAE:

1) T is invertible in $\mathcal{B}(X)$ 2) $\text{Ker } T = \{0\}$ 3) $\text{Ran } T = X$

1) T^* is invertible in $\mathcal{B}(X)$ 2) $\text{Ker } T^* = \{0\}$ 3) $\text{Ran } T^* = X^*$

Proof: We will prove $2 \Rightarrow 3 \Rightarrow 2' \Rightarrow 3' \Rightarrow 2$, $1 \Rightarrow 1' \Rightarrow 2'$, $2 \& 3 \Rightarrow 1$

(2) $\Rightarrow$ (3): If $T^k X = T^{k+1} X$ for some k , we are done by the lemma.

Define $X_k := T^k X$, $k \geq 0$, and note that $X_0 \supset X_1 \supset X_2 \supset X_3 \dots$

Assume that all inclusions are strict, i.e. $X_k \supsetneq X_{k+1} \forall k$.

The subspaces X_k are closed by the closed range lemma (by induction). By the almost orthogonality lemma: $\exists \{y_k\}_0^\infty$ s.t.

i) $y_k \in X_k \forall k$ ii) $\|y_k\| = 1 \forall k$ iii) $\text{dist}(y_k, X_{k+1}) \geq \frac{1}{2}$

Since K is compact $\{K y_k\}$ contains a convergent subsequence $\{K y_{k_j}\}_{j=1}^\infty$. On the other hand, if $j < m$

$$\begin{aligned} K y_{k_j} - K y_{k_m} &= (K y_{k_j} - y_{k_j}) - (K y_{k_m} - y_{k_m}) + y_{k_j} - y_{k_m} \\ &= y_{k_j} - y_{k_m} - \underbrace{(T y_{k_j} - T y_{k_m})}_{\in X_{k_j+1}} \end{aligned}$$

Since $y_{k_m} \in X_{k_m}$, $j < m$, $k_j < k_m$, $k_j+1 \leq k_m$, $X_{k_m} \subset X_{k_j+1}$

$\Rightarrow y_{k_m} \in X_{k_j+1}$; $T y_{k_m} \in T X_{k_j+1} \subset X_{k_j+1}$; $T y_{k_j} \in T(X_{k_j}) = X_{k_j+1}$

$\Rightarrow K y_{k_j} - K y_{k_m} = y_{k_j} + R$, $R \in X_{k_j+1}$. Since $\|y_{k_j} + R\| \geq \frac{1}{2}$ by (iii),

we get a contradiction ($\{K y_{k_j}\}$ is not Cauchy).

(3) $\Rightarrow$ (2): Take $\phi \in \text{Ker } T^*$. We have

$$T^* \phi = 0 \Leftrightarrow \langle x, T^* \phi \rangle = 0 \forall x \in X \Leftrightarrow \langle T x, \phi \rangle = 0 \forall x \in X \Leftrightarrow$$

$$\Leftrightarrow \langle y, \phi \rangle = 0 \forall y \in X \text{ (by 3)} \Leftrightarrow \phi = 0$$

(2') $\Rightarrow$ (3'): (the same as $2 \Rightarrow 3$ using Schauder's theorem)

(3') $\Rightarrow$ (2): Assume that $\text{Ran } T^* = X^*$ and take $x \in \ker T$.

We have $Tx = 0 \Leftrightarrow \langle Tx, \phi \rangle = 0 \ \forall \phi \in X^* \Leftrightarrow \langle x, T^*\phi \rangle \ \forall \phi \in X^* \Leftrightarrow \langle x, \psi \rangle = 0 \ \forall \psi \in X^*$ (by 3') $\Leftrightarrow x = 0$ lemma on sufficient amount of functionals

Conclusion: $(2) \Leftrightarrow (3) \Leftrightarrow (2') \Leftrightarrow (3')$

(1) $\Rightarrow$ (1'): We already know for arbitrary $T \in \mathcal{B}(X)$.

(1') $\Rightarrow$ (2'): Invertible operators are injective.

$(2') \Leftrightarrow (2), (3)$

(2 & 3) $\Rightarrow$ (1): This holds for every $T \in \mathcal{B}(X)$ by the following fundamental theorem from general F.A.:

Theorem [linear mapping theorem]: Let X, Y be a Banach space, $T: X \rightarrow Y$ - linear bijection. Then $T \in \mathcal{B}(X, Y) \Leftrightarrow T^{-1} \in \mathcal{B}(Y, X)$.

Check injectivity: $Tx_1 = Tx_2 \Leftrightarrow T(x_1 - x_2) = 0 \Leftrightarrow x_1 - x_2 = 0$ (by 4) $\Leftrightarrow x_1 = x_2$
surjectivity: $\text{Ran } T = X$ (by 3) ▣

Remark: If $T = I - K$, $K \in \mathcal{S}_\infty(X)$, and $\ker T = \{0\}$ then the equation $Tx = y$ has a unique solution for every $y \in X$.

Remark: $\ker T = \{0\} \Leftrightarrow \ker T^* = \{0\}$, so we have proved half of Fredholm Alternative.

Theorem: Let X be a Banach space, $T \in \mathcal{B}(X)$. Then $\overline{TX} = \{x \in X \mid \langle x, \phi \rangle = 0 \ \forall \phi \in \ker T^*\}$.

Remark: This implies the other half of Fredholm alternative since $\overline{TX} = TX$ for $T = I - K$, $K \in \mathcal{S}_\infty(X)$.

Lemma [separation lemma]: Let Y be a Banach space, $Y_0 \subsetneq Y$ - a closed subspace, then $\exists \phi \in Y^* : \phi|_{Y_0} = 0, \phi(y) \neq 0$ for some $y \in Y \setminus Y_0$.

Proof: Take $y \in Y \setminus Y_0$, define $\phi_0 : \text{span}\{y, Y_0\} \rightarrow \mathbb{C}$ by $\phi_0(cy + y_0) \mapsto c$, for $c \in \mathbb{C}, y_0 \in Y_0$.

$$1) \{cy + y_0 \mid c \in \mathbb{C}, y_0 \in Y_0\} = \text{span}\{y, Y_0\} \text{ - clear } \checkmark$$

$$2) \underbrace{cy + y_0}_e = \underbrace{\tilde{c}y + \tilde{y}_0}_{\tilde{e}} \Leftrightarrow (c - \tilde{c})y = y_0 - \tilde{y}_0 \in Y_0 \Leftrightarrow c = \tilde{c} \Rightarrow \phi(e) = \phi(\tilde{e})$$

$\Rightarrow$ correctness ok

3) ϕ_0 is linear - clear

$$4) \phi_0|_{Y_0} = 0 \quad (c=0 \text{ on } Y)$$

$$5) |\phi_0(cy + y_0)| \stackrel{!}{\leq} A \|cy + y_0\| \quad \forall c, y_0$$

$$|\phi_0(cy + y_0)| = |c| = \left(\underbrace{\text{dist}(y, Y_0)}_{\text{closed}} \right)^{-1} \cdot |c| \cdot \text{dist}(y, Y_0)$$

$$= \text{dist}(y, Y_0)^{-1} \cdot \text{dist}(|c|y, Y_0) = \text{dist}(y, Y_0)^{-1} \cdot \text{dist}(cy, Y_0)$$

$$|c|y = d \cdot e \cdot y_0, |d|=1$$

$$\text{dist}(|c|y, Y_0) = \text{dist}\left(\underbrace{d}_{\mathbb{C}} \underbrace{|c|y_0}_{Y_0}, \underbrace{Y_0}_Y\right)$$

$$\leq \text{dist}(y, Y_0)^{-1} \|cy + y_0\|, \quad \text{so } A = \text{dist}(y, Y_0)^{-1} \text{ works}$$

$$6) \phi_0(y) = 1 \neq 0$$

$\Rightarrow$ Use the Hahn-Banach theorem and extend ϕ_0 to the whole Y . □

We actually proved that the lemma holds $\forall y \in Y \setminus Y_0$.

We can now prove the theorem from above.

Theorem: Let X be a Banach space, $T \in \mathcal{B}(X)$. Then $\overline{TX} = \{x \in X \mid \langle x, \phi \rangle = 0 \ \forall \phi \in \ker T^*\}$.

Proof: We have $TX \subset E$, $E = \{x \mid \langle x, \phi \rangle = 0 \ \forall \phi \in \ker T^*\}$, because $\langle Tx, \phi \rangle = \langle x, T^*\phi \rangle = 0 \ \forall \phi \in \ker T^*$.

Then $\overline{TX} \subset \overline{E} = E$ since E is closed.

$\left(\begin{array}{l} x_n \rightarrow x \text{ and } \langle x_n, \phi \rangle = 0 \text{ for some } \phi \in X^* \\ \text{then } \langle x, \phi \rangle = \lim \langle x_n, \phi \rangle = 0 \end{array} \right)$

We now need to check $\overline{TX} \supset X$. If not, the inclusion $\overline{TX} \subset E$ is proper and by the separation lemma $\exists \phi: \phi|_{\overline{TX}} \stackrel{(*)}{=} 0$ but $\phi(e) \neq 0$ for some $e \in E$.

$(*) \Rightarrow \phi|_{TX} = 0 \Leftrightarrow \langle Tx, \phi \rangle = 0 \ \forall x \in X \Leftrightarrow \langle x, T^*\phi \rangle = 0 \ \forall x \in X$
 $\Leftrightarrow \phi \in \ker T^* \Rightarrow \phi(e) = 0 \ \forall e \in E$ by definition of E , contradiction. $\square$

Classical Form of Fredholm alternative for integral equations

Theorem: Let (S, μ) be a space with measure μ , and let $K(x, y): S \times S \rightarrow \mathbb{C}$:

$$\iint_{S \times S} |K(x, y)|^2 d\mu(x) d\mu(y) < \infty.$$

Then either equation $f(y) + \int_S K(x, y) f(x) d\mu(x) \stackrel{(**)}{=} 0$ has only the trivial solution $f=0$ and the equation

$$f(y) + \int_S K(x, y) f(x) d\mu(x) = g(y)$$

is solvable for every $g \in L^2(S, \mu)$ or the equation $(**)$ has a non-trivial solution in $L^2(S, \mu)$.

Remark: Uniqueness implies existence.

Proof: Let's define $T = I - K$, $(Kf)(y) = \int_S K(x, y) f(x) d\mu(x)$. Consider T as an operator on $L^2(S, \mu)$.

Then $(**) \Leftrightarrow \ker T = \{0\} \Leftrightarrow TL^2(S, \mu) = L^2(S, \mu)$ by Fredholm alternative, modulo the fact that $K \in S_\infty(L^2(S, \mu))$. $\square$

↑ This we postpone until Hilbert spaces theory.

Remarks: Further reading:

1) $\dim \ker T = \dim \ker T^*$ if $T = I - K$, $K \in S_\infty$, it coincides with the dimension of the space of solutions

2) There is a version of Fredholm theory for general operators $T \in \mathcal{B}(X)$: $\dim(\ker T) < \infty$, $\dim(X/\operatorname{Ran} T) < \infty$.

Spectrum of compact operators

October 29, 2015

Definition: Let $T \in \mathcal{B}(X)$, X Banach space.

$\sigma(T) := \{\lambda \in \mathbb{C} \mid \lambda I - T \text{ is not invertible in } \mathcal{B}(X)\}$

is called the **spectrum** of T .

Definition: A number $\lambda \in \mathbb{C}$ is called an **eigenvalue** of T if $\exists e \in X \setminus \{0\}$. $Te = \lambda e$.

Definition: $\sigma_p(T) := \{\lambda \text{ eigenvalue of } T\} \dots$ **point spectrum** of T .

Remark: We have $\sigma_p(T) \subset \sigma(T)$ for every $T \in \mathcal{B}(X)$.

Proof: $\lambda \in \sigma_p(T) \Rightarrow \lambda I - T$ is not injective because $\ker(\lambda I - T) \neq \{0\}$.

Remark: In general, we might have $\sigma_p(T) \neq \sigma(T)$ and even $\sigma_p(T) = \emptyset$.

Theorem: Let X be a Banach space, $\dim X = +\infty$, and let $K \in S_{\infty}(X)$. Then $0 \in \sigma(K)$, $\sigma(K) \setminus \{0\} \subset \sigma_p(K)$, moreover, each eigenvalue has a finite multiplicity, and $\#\{\lambda \in \sigma_p(K) \mid |\lambda| \geq r\} < \infty$ for every $r > 0$.

Proof: $0 \in \sigma(K)$ since $0 \notin \sigma(K)$, then $\exists K^{-1} \in B(X)$:
 $I = \underbrace{K}_{S_{\infty}(X)} \underbrace{K^{-1}}_{B(X)} = K^{-1}K$, but then $I \in S_{\infty}(X) \Rightarrow \dim X < \infty$, contradiction.

Now let's prove that $\sigma(K) \setminus \{0\} \subset \sigma_p(K)$.

Take $\lambda \in \sigma(K)$ and assume that $\lambda \in \sigma_p(K)$. Then

$$\text{Ker}(\lambda I - K) = \{0\} \Leftrightarrow \text{Ker}(I - \frac{1}{\lambda}K) = \{0\}$$

$\Rightarrow I - \frac{1}{\lambda}K$ is invertible by Fredholm theorem - contradiction.

Now let's prove that $\dim E_{\lambda} < \infty$.

$$E_{\lambda} := \{e \in X \mid Ke = \lambda e\} \quad \left(\begin{array}{l} \text{definition for:} \\ \lambda \text{ has finite multiplicity} \end{array} \right)$$

If this is not the case, there is a sequence $\{e_n\}_{n=1}^{\infty}$ such that $\text{dist}(e_n, \text{span}\{e_1, \dots, e_{n-1}\}) \geq 1/2$, $\|e_n\| = 1$.

Consider $\{Ke_n\} = \{\lambda e_n\}$: we cannot choose a convergent subsequence from this sequence - contradiction with $K \in S_{\infty}(X)$.

To prove $\#\{\lambda \in \sigma_p(K) \mid |\lambda| \geq r\} < \infty$ for every $r > 0$, assume the converse and let $e_n \in X$: $Ke_n = \lambda_n e_n$, $\|e_n\| = 1$, $\lambda_n \in \sigma_p(K)$, $\lambda_n \neq \lambda_k$ for $k \neq n$.

Define $E_n := \text{span}\{e_1, \dots, e_n\} \forall n$.

Observation: $E_{n+1} \supsetneq E_n \forall n$.

Clearly $E_{n+1} \supset E_n$. If $E_{n+1} = E_n$ for some n , there exists the first such n . Then

$$\lambda_{n+1} e_{n+1} = \sum_{k=1}^n \lambda_{n+1} \alpha_k e_k$$

$$\lambda_{n+1} e_{n+1} = \sum_{k=1}^n d_k \lambda_k e_k$$

$$\Rightarrow 0 = \sum_{k=1}^n d_k (\lambda_{n+1} - \lambda_k) e_k \Rightarrow d_{n+1} - \lambda_n = 0$$

$$\Rightarrow \lambda_{n+1} = \lambda_n \dots \text{contradiction}$$

$\Rightarrow$ The observation is true.

Let's choose $y_{n+1} \in E_{n+1}$ such that $\|y_{n+1}\| = 1$, $\text{dist}(y_{n+1}, E_n) \geq \frac{1}{2}$.
It remains to prove that $\{K y_n\}$ does not have a convergent subsequence.

$$y_{n+1} = d_{n+1} e_{n+1} + \tilde{R}_n, \text{ where } \tilde{R}_n \in E_n$$

$$K y_{n+1} - K y_{m+1} = \lambda_{n+1} d_{n+1} e_{n+1} + \tilde{R}_n - \lambda_{m+1} d_{m+1} e_{m+1} - \tilde{R}_m$$

Assume that $n \geq m+1$, then

$$\underbrace{\tilde{R}_n}_{\hat{E}_n} - \underbrace{\lambda_{m+1} d_{m+1} e_{m+1}}_{\hat{E}_{m+1} \subset E_n} + \underbrace{\tilde{R}_m}_{\hat{E}_m \subset E_n} \in E_n$$

$$\begin{aligned} \Rightarrow \|K y_{n+1} - K y_{m+1}\| &\geq \text{dist}(\lambda_{n+1} d_{n+1} e_{n+1}, E_n) \\ &= |\lambda_{n+1}| \text{dist}(d_{n+1} e_{n+1}, E_n) \\ &= |\lambda_{n+1}| \text{dist}(y_{n+1}, E_n) \\ &\geq r \cdot \frac{1}{2} > 0 \end{aligned}$$



Scheme of the solution of Dirichlet problem in $\mathbb{R}^n$, $n \geq 3$, by means of Fredholm theory

Dirichlet problem: Find $u \in C^2(\Omega) \cap C(\bar{\Omega})$ (Ω is a domain in $\mathbb{R}^n$, $\partial\Omega \in C^2$) such that

$$\begin{cases} \Delta u = 0 \\ u|_{\partial\Omega} = f \end{cases} \quad \text{where } f \in C(\partial\Omega)$$



Physics interpretation: heat

Scheme for solution: we search for a solution of the form

$$u(x) = \int_{\partial\Omega} K(x,y) \varphi(y) dy, \quad K(x,y) = c_n \frac{(x-y, n_y)}{\|x-y\|_{\mathbb{R}^n}^n}$$

n_y is the outward unit normal, $c_n \in \mathbb{R}$

$\Delta u = 0$ For every $\varphi \in C(\partial\Omega)$, we only need to find good φ (such that (*) will hold)

$$u_\varphi(x) = -\varphi(x) + \int_{\partial\Omega} K(x,y) \varphi(y) dy \quad \text{if } x \in \partial\Omega$$

$$u_\varphi = (-I + K)\varphi, \quad K\varphi = \int_{\partial\Omega} K(x,y) \varphi(y) dy$$

To check that $\exists \varphi: u_\varphi = f$ on $\partial\Omega$ we just check that $K \in S_\infty(C(\partial\Omega))$ and $\text{Ker}(-I + K) = \{0\}$
 $\Rightarrow$ we are done by Fredholm alternative.

This is hard to prove (course on PDEs)

PART II Banach Algebras

November 4, 2025

Definition: A is a **Banach algebra** if A is a Banach space with the operation of multiplication such that:

$$1) (xy)z = x(yz); \quad x, y, z \in A$$

$$2) (x+y)z = xz + yz$$

$$x(y+z) = xy + xz$$

$$3) (dx)y = x(dy); \quad d \in \mathbb{C}$$

$$4) \exists e \in A. \quad xe = ex = x \quad \forall x \in A.$$

$$5) \|xy\| \leq \|x\| \cdot \|y\| \quad \forall x, y \in A.$$

Examples: 1) X Banach space $\Rightarrow \mathcal{B}(X)$ is a Banach algebra with unity $e = I$ and norm $\|T\| = \sup_{\|x\| \leq 1} \|Tx\|$

2) **Calkin algebra:** $\mathcal{B}(X)/\mathcal{S}_\infty(X)$, $e = I + \mathcal{S}_\infty(X)$

$$\|T\|_{\mathcal{B}(X)/\mathcal{S}_\infty(X)} = \inf_{K \in \mathcal{S}_\infty(X)} \|T - K\| = \text{dist}(T, \mathcal{S}_\infty(X))$$

3) K -compact Hausdorff space, $C(K)$ is a Banach algebra, $e = 1$, $\|f\|_{C(K)} = \max_{x \in K} |f(x)|$
 $f: K \rightarrow \mathbb{C}$

4) **$\mathcal{W}(\mathbb{T})$ - Wiener algebra** on $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$.

$$\| \{ f = \sum_{k \in \mathbb{Z}} c_k z^k \mid c_k \in \mathbb{C}, \sum |c_k| < \infty \}$$

$$\|f\|_{\mathcal{W}(\mathbb{T})} = \sum_{k \in \mathbb{Z}} |c_k|, \quad e = 1, \quad \begin{array}{l} \text{multiplication is the usual} \\ \text{multiplication of functions} \end{array}$$

$$\|f \cdot g\|_{\mathcal{W}(\mathbb{T})} \leq \|f\|_{\mathcal{W}(\mathbb{T})} \cdot \|g\|_{\mathcal{W}(\mathbb{T})}$$

$\uparrow$ exercise

$$5) L^\infty(\mathbb{R}), \quad \|f\| = \operatorname{ess\,sup}_{\mathbb{R}} |f|$$

6) $H^\infty(\mathbb{D})$ - set of bounded analytic functions on $\mathbb{D} := \{|z| < 1\}$

$$\|f\|_{H^\infty(\mathbb{D})} = \sup_{|z| < 1} |f(z)|$$

Remark: 1) & 2) are noncommutative Banach algebras, others are commutative.

Remark: If A is a Banach space with multiplication and properties 1), 2), 3), 5), then we can always add the identity to convert A to a Banach algebra as follows:

$$\mathcal{A} = A \times \mathbb{C}, \quad (x, \alpha) + (y, \beta) = (x + y, \alpha + \beta)$$

$$\gamma(x, \alpha) = (\gamma x, \gamma \alpha)$$

$$(x, \alpha) \cdot (y, \beta) = (xy + \alpha y + \beta x, \alpha \beta)$$

$$("(\alpha + \alpha e)(y + \beta e)" = xy + \alpha y + \beta x + \alpha \beta e) \quad e = (0, 1)$$

$$\|(x, \alpha)\| = \|x\| + |\alpha| \quad \leftarrow ! \text{ corrected on Nov. 5th}$$

$\Rightarrow \mathcal{A}$ is a Banach algebra with identity and $(A, 0) \subset \mathcal{A}$.

Example: $L^1(\mathbb{R})$, multiplication $f * g = \int_{\mathbb{R}} f(y)g(x-y)dy$

$\|f\|_{L^1(\mathbb{R})} = \int_{\mathbb{R}} |f| dx$ - this is a Banach algebra without identity, and the above construction is equivalent to consideration of $\mathcal{A} = \mathbb{C}\delta_0 + L^1(\mathbb{R})$

$$\left(\begin{array}{l} \text{measure} \\ \delta_0(s) = \begin{cases} 1; & 0 \in s \\ 0; & 0 \notin s \end{cases}, \quad \delta_0 * f = f * \delta_0 = \int f(y)\delta_0(x-y)dy = f(x) \\ \langle f, \delta_0 \rangle = f(0) = \int f d\delta_0, \quad f \in C(\mathbb{R}) \end{array} \right) \quad \text{measures} \rightarrow \text{no } dy$$

Definition: A Banach algebra, $x \in A$. We say $x \in A$ is invertible if $\exists x^{-1} \in A$. $xx^{-1} = x^{-1}x = e$.

If x is invertible, then x^{-1} is unique.

Definition: $G(A) := \{x \in A \mid x \text{ is invertible}\}.$

↑ we use G , because $G(A)$ is a group

Definition: $x \in A$. $\delta(x) := \{\lambda \in \mathbb{C} \mid \lambda e - x \text{ is not invertible}\}.$

Example: If A is the set of $n \times n$ matrices with complex coefficients, then for $T \in A$, $\delta(T)$ is the set of eigenvalues (the usual spectrum of the matrix).

Example: $A = \mathbb{C}(K)$, $\sigma(f) = ?$

$\lambda \in \mathbb{C}$: $\lambda - f$ is not invertible in $\mathbb{C}(K)$.

Since $g \in G(\mathbb{C}(K)) \Leftrightarrow \frac{1}{g} \in \mathbb{C}(K)$, we have $\lambda - f \in G(\mathbb{C}(K)) \Leftrightarrow \frac{1}{\lambda - f} \in \mathbb{C}(K) \Leftrightarrow \lambda \notin f(K)$

$\Rightarrow \sigma(f) = f(K) \leftarrow$ compact non-empty subset of $\mathbb{C}$

Basic properties

Proposition 1: Let A be a Banach algebra. Then $\|e\| \geq 1$.

Proof: $\|e\| = \|e \cdot e\| \leq \|e\| \cdot \|e\| \Rightarrow \|e\| = 0$ or $\|e\| \geq 1$
 $\downarrow$
 $e = 0 \quad \times$

Definition: A Banach algebra is called **unital** if $\|e\| = 1$.

Proposition 2: If A is an arbitrary Banach algebra, then the algebra $A \times \mathbb{C}$ is unital.

Proof: $\|(0, 1)\| = \|0\| + \|1\| = 1$

↑
!Note: This was wrong previously and corrected on November 5th.

⚠ From now on, all Banach algebras are unital.

Proposition 3: If $\begin{matrix} x_n \rightarrow x \\ y_n \rightarrow y \end{matrix}$ in A , then $x_n \cdot y_n \rightarrow xy$.

Proof: $\|x_n \cdot y_n - xy\| \leq \underbrace{\|x_n - x\|}_{\rightarrow 0} \underbrace{\|y_n\|}_{\text{bdd}} + \|x\| \underbrace{\|y_n - y\|}_{\rightarrow 0} \rightarrow 0$ □

Proposition 4: Let $a \in A$, $\|a\| \leq 1$, then $e - a \in G(A)$.

Proof: Define $(e - a)^{-1} = e + a + a^2 + \dots = \sum_{k=0}^{\infty} a^k$ this series converges because A_n is a Banach space and $\sum \|a^k\| \leq \sum \|a\|^k < \infty$

Let us check that $(e - a)^{-1}(e - a) = e$

$$\Leftrightarrow \lim_{n \rightarrow \infty} \left(\sum_{k=0}^n a^k \right) (e - a) = e$$

$$e - \underbrace{a^{n+1}}_{\rightarrow 0} \rightarrow e, \text{ since } \|a^{n+1}\| \leq \|a\|^{n+1} \rightarrow 0$$

Similarly, $(e - a)(e - a)^{-1} = e$.

Proposition 5: $G(A)$ is open in A .

Proof: Let $a \in G(A)$, $b \in A$, then $a - b = \underbrace{a}_{\in G(A)} \underbrace{(e - a^{-1}b)}_{\in G(A) \text{ if } \|a^{-1}b\| < 1}$
 $\|a^{-1}b\| < 1$ holds for all b : $\|b\| \leq \frac{1}{\|a^{-1}\|}$

$\Rightarrow B(a, \frac{1}{\|a^{-1}\|}) \subset G(A) \Rightarrow G(A)$ is open.

Proposition 6: Let $x \in G(A)$, $x_n \rightarrow x$ in A . Then $x_n \in G(A)$ for n large enough, and $x_n^{-1} \rightarrow x^{-1}$ as $n \rightarrow \infty$.

Proof: Write $x_n = x + z_n$, we have $x_n \in G(A)$ for n large enough by Proposition 5.

$$(x + z_n)^{-1} - x^{-1} = (x(e + x^{-1}z_n))^{-1} - x^{-1} = \underbrace{(e + x^{-1}z_n)^{-1}}_{y_n} \cdot x^{-1} - x^{-1} \xrightarrow[\text{Prop 3}]{\uparrow} ex^{-1} - x^{-1} = 0$$

$$y_n \rightarrow e, \text{ see Prop. 4: } \|y_n - e\| \leq \sum_{k=1}^{\infty} \|x^{-1} \cdot z_n\|^k \rightarrow 0 \quad \|z_n\| \rightarrow 0$$

□

Theorem: Let A be a Banach algebra, $a \in A$. Then $\sigma(A)$ is a nonempty compact subset of $\mathbb{C}$.

Proof: $\sigma(a) = \mathbb{C} \setminus \rho(a)$, $\rho(a) := \{\lambda \mid \lambda e - a \text{ is invertible}\}$
resolvent set

$\rho(a)$ is open by Prop. 5 $\Rightarrow \sigma(a)$ is closed.

Let's check that $\sigma(a)$ is bounded: $\sigma(a) \subset \{\lambda \mid |\lambda| \leq \|a\|\}$
 $\Leftrightarrow \rho(a) \supset \{\lambda \mid |\lambda| \geq \|a\|\}$.

Take $\lambda: |\lambda| \geq \|a\|$, then $\lambda e - a = \lambda e \underbrace{\left(e - \frac{a}{\lambda}\right)}_{\text{invertible by Prop. 4}}$

It remains to check that $\sigma(a) \neq \emptyset$.

Assume that $\sigma(a) = \emptyset$ and consider the function

$$f_\phi(\lambda) = \phi((\lambda e - a)^{-1}) \text{ for some } \phi \in A^*; \lambda \in \mathbb{C}$$

Let's check that f_ϕ is analytic. Take $\lambda_0 \in \mathbb{C}$, consider

$$\lim_{\lambda \rightarrow \lambda_0} \frac{f_\phi(\lambda) - f_\phi(\lambda_0)}{\lambda - \lambda_0} = \lim_{\lambda \rightarrow \lambda_0} \phi \left(\frac{(\lambda e - a)^{-1} - (\lambda_0 e - a)^{-1}}{\lambda - \lambda_0} \right)$$

$$[x^{-1} - y^{-1} = x^{-1}(y - x)y^{-1}] = \lim_{\lambda \rightarrow \lambda_0} \phi \left(\frac{(\lambda e - a)^{-1}((\lambda e - a) - (\lambda_0 e - a))(\lambda_0 e - a)^{-1}}{\lambda - \lambda_0} \right)$$

$$= \lim_{\lambda \rightarrow \lambda_0} \phi \left((\lambda e - a)^{-1} (\lambda_0 e - a)^{-1} \right)$$

ϕ continuous,
continuity of
multiplication

$\Rightarrow (\lambda_0 e - a)^{-1}$ Prop. 6

$$= -\phi((\lambda_0 e - a)^{-2})$$

holomorphic

$\Rightarrow f_\phi$ is analytic (f_ϕ is Hol($\mathbb{C}$))

Take $\lambda: |\lambda| \geq 2\|a\|$, $|f_\phi(\lambda)| = \|\phi\| \cdot \|(\lambda e - a)^{-1}\| = \|\phi\| \underbrace{|\lambda|^{-1}}_{(*)} \underbrace{\|(e - \frac{a}{\lambda})^{-1}\|}_{\text{uniformly bounded for } \lambda: |\lambda| \geq 2\|a\|}$

f_ϕ is bdd on $\mathbb{C}$ by the max. principle

$\Rightarrow f_\phi = c_\phi \in \mathbb{C}$, but $c_\phi = 0$ by (*)

$\Rightarrow \phi(\lambda e - a) = 0 \forall \lambda \in \mathbb{C}, \forall \phi \in A^* \Rightarrow \lambda e - a = 0 \forall \lambda \in \mathbb{C}$
 $\leadsto$ contradiction



November 5, 2025

Definition: Let A_1, A_2 be Banach algebras.

We say that A_1 is **isomorphic** to A_2 ($A_1 \cong A_2$) if there exists a map $j: A_1 \rightarrow A_2$:

$$j(\alpha a + \beta b) = \alpha j(a) + \beta j(b) \quad (*)$$

$$j(ab) = j(a) \cdot j(b) \quad (**)$$

$$\|j(a)\|_{A_2} = \|a\|_{A_1}$$

j is bijective

Theorem [Banach-Mazur]: Let A be a Banach algebra s.t. $G(A) = A \setminus \{0\}$. Then $A \cong \mathbb{C}$.

Proof: For every $a \in A$ we have $\sigma(a) \neq \emptyset$. So there is a $\lambda(a) \in \mathbb{C}$:
 $\lambda(a)e - a$ is not invertible $\Leftrightarrow \lambda(a)e - a = 0 \Leftrightarrow a = \lambda(a)e$.
 In particular, such $\lambda(a)$ is unique.

Define $j: A \rightarrow \mathbb{C}$, $a \mapsto \lambda(a)$. We have:

$$\left. \begin{aligned} a+b &= \lambda(a+b)e \\ a+b &= \lambda(a)e + \lambda(b)e \end{aligned} \right\} \Rightarrow \lambda(a+b) = \lambda(a) + \lambda(b)$$

$$\text{Similarly, } \lambda(\alpha a) = \alpha \lambda(a), \lambda(ab) = \lambda(a) \cdot \lambda(b) \Rightarrow (*), (**) \quad \checkmark$$

$$\|j(a)\|_{\mathbb{C}} = \|a\|_A \Leftrightarrow |\lambda(a)| = \|a\|_A \Leftrightarrow \|\lambda(a)e\|_A = \|a\|_A \quad \checkmark$$

$$\left. \begin{aligned} j(a) = j(b) &\Leftrightarrow \lambda(a)e = \lambda(b)e \Leftrightarrow a = b \\ j(e) = 1 &\Rightarrow j(A) \supset j(\mathbb{C} \cdot e) = \mathbb{C} \end{aligned} \right\} j \text{ is a bijection}$$

Definition: Let A be a Banach algebra, then $r(a) := \sup\{|\lambda|, \lambda \in \sigma(a)\}$ is called the **spectral radius** of a .



Theorem: Let A be a Banach algebra. Then

$$r(a) = \inf_{n \geq 1} \|a^n\|^{1/n} = \lim_{n \rightarrow \infty} \|a^n\|^{1/n}.$$

In particular, the limit above exists.

Proof: Step 1: $r(a) \leq \|a\|$, because

$$\lambda e - a = \lambda \underbrace{\left(e - \frac{a}{\lambda}\right)}_b \quad \|b\| < 1 \text{ for } \lambda: |\lambda| \geq \|a\|$$

invertible element in A (Prop. 4)

$$\Rightarrow \lambda e - a \in G(A) \Rightarrow \lambda \in \rho(a) \Rightarrow \sigma(a) \subset \mathbb{B}[0, \|a\|]. \quad \checkmark$$

Step 2: $r(a) \leq \|a^n\|^{1/n}$ for every $n \geq 2$.

Take $\lambda \in \mathbb{C}$, and consider

$$\lambda^n e - a^n = (\lambda e - a) \cdot p_\lambda(a), \quad p_\lambda \text{ is a polynomial}$$

If $\lambda \in \sigma(a)$, then $\lambda^n e - a^n = z_1 \cdot z_2$, where $z_1, z_2 \in A$, $z_1 \notin G(A)$, $z_1 z_2 = z_2 z_1$. If $z = \lambda^n e - a^n$ is invertible, $\exists z^{-1} \in A$.

$$\begin{aligned} z^{-1} z_1 z_2 &= z_1 (z_2 z^{-1}) = e \\ \parallel & \\ (z^{-1} z_1) z_2 &= e \Rightarrow z_2 \in G(A) \dots \text{contradiction} \end{aligned}$$

$$\Rightarrow \lambda^n \in \sigma(a^n) \xrightarrow{\text{Step 1}} |\lambda^n| \leq \|a^n\| \Rightarrow |\lambda| \leq \|a^n\|^{1/n}. \quad \checkmark$$

$$\text{Step 3: } \inf_{n \geq 1} \|a^n\|^{1/n} \leq \liminf_{n \rightarrow \infty} \|a^n\|^{1/n} \leq \overline{\lim}_{n \rightarrow \infty} \|a^n\|^{1/n} \leq r(a) \leq \inf_{n \geq 1} \|a^n\|^{1/n}$$

$\hookrightarrow$ This implies the claim.

Notation note: $\overline{\lim} = \limsup$, $\underline{\lim} = \liminf$.

All that remains is $\overline{\lim}_{n \rightarrow \infty} \|a^n\|^{1/n} \leq r(a)$. Take $\phi \in A^*$, $\|\phi\| \leq 1$, $f_\phi(\lambda) := \phi((\lambda e - a)^{-1})$ for $\lambda \in \rho(a)$ (this is an analytic function on $\rho(a)$)

$$\varepsilon > 0: \quad \frac{1}{2\pi i} \oint_{|\lambda|=r(a)+\varepsilon} \lambda^k f_\phi(\lambda) d\lambda = \frac{1}{2\pi i} \int_{|\lambda|=2\|a\|} \lambda^k f_\phi(\lambda) d\lambda =$$

$$= \frac{1}{2\pi i} \int_{|\lambda|=2\|a\|} \lambda^k \phi \left(\lambda \sum_{n=0}^{\infty} \frac{a^n}{\lambda^{n+1}} \right) d\lambda = \frac{1}{2\pi i} \int_{|\lambda|=2\|a\|} \lambda^{k+1} \sum_{n=0}^{\infty} \frac{\phi(a^n)}{\lambda^{n+1}} d\lambda$$

$$= \phi(a^{k+2}) \quad (\text{Cauchy formula from complex analysis})$$

$$\Rightarrow \phi(a^{k+2}) \leq \left| \frac{1}{2\pi i} \oint_{|\lambda|=r(a)+\varepsilon} \lambda^k f_\phi(\lambda) d\lambda \right|$$

$$\leq \max_{|\lambda|=r(a)+\varepsilon} (|\lambda|^k \cdot |f_\phi(\lambda)|) \frac{1}{2\pi} (2\pi(r(a)+\varepsilon))$$

$$= (r(a) + \varepsilon)^{k+1} \underbrace{\|\phi\|}_{\leq 1} \cdot \underbrace{\sup_{|\lambda|=r(a)+\varepsilon} \|(\lambda e - a)^{-1}\|}_{\text{constant depending only on } \varepsilon, \text{ because } |\lambda|=r(a)+\varepsilon \text{ is a compact set and } \lambda \mapsto \|(\lambda e - a)^{-1}\| \text{ is continuous}}$$

$$\sup_{\|\phi\| \leq 1} |\phi(a^{k+2})| = \|a^{k+2}\|$$

$$\frac{1}{C_\varepsilon} \cdot (r(a) + \varepsilon)^k$$

$$\Rightarrow \overline{\lim} \|a^k\|^{1/k} \leq r(a) + \varepsilon \quad \text{for every } \varepsilon > 0.$$

