

Introduction to algebraic geometry

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I. AFFINE VARIETIES (A fine raznoterosti)

1. Recap of basic notions

A polynomial over a ring R is a formal expression $p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0$, where $a_i \in R$, $n = \text{degree of } p$ (if $a_n \neq 0$). The set of all polynomials in x over R is denoted $R[x]$. All rings will be commutative and with 1. $R[x]$ is a ring for usual addition and multiplication of polynomials. Suppose R is a subring of a ring S and $a \in S$. Then we can compute $a_n a^n + a_{n-1} a^{n-1} + \dots + a_0 \in S$. We denote this element $p(a)$ and call it the value of p at a . If $p(a) = 0$, then a is a root/zero of p . We have a polynomial function $S \rightarrow S, a \mapsto p(a)$.

Polynomial in two variables x and y with coefficients in a ring R is an expression $p(x, y) = \sum_{i=0}^n \sum_{j=0}^m a_{ij} x^i y^j$, where $a_{ij} \in R$. The expression $a_{ij} x^i y^j$ is called a monomial. The degree of a monomial $x^i y^j$ is $i+j$. The degree of the polynomial $p(x, y) = \max \{i+j \mid a_{ij} \neq 0\}$. The set of all polynomials in two variables is a ring. We denote it by $R[x, y]$.

$$R[x, y] \cong R[x][y] \cong R[y][x].$$

Similarly we define polynomials in more variables.

$$R[x_1, \dots, x_n] \cong R[x_1, \dots, x_{n-1}][x_n].$$

If R is a subring of S and $p \in R[x_1, \dots, x_n]$ and $a \in (a_1, \dots, a_n) \in S^n$, we can compute $p(a_1, \dots, a_n) \in S$. We get a function $S^n \rightarrow S, a \mapsto p(a)$.

Let R be a ring. An ideal of R is a subset $I \subseteq R$ s.t.

i) If $x, y \in I$, then $x+y \in I$.

ii) If $a \in R$ and $x \in I$, then $ax \in I$.

$I \triangleleft R$.

If an ideal I contains an invertible element, then $I=R$.

If $M \subseteq R$ is some set, then

$(M) := \left\{ \sum_{i=0}^n a_i x_i \mid n \in \mathbb{N}, a_i \in R, x_i \in M \right\}$ is an ideal.

We call it the ideal generated by M . If $M = \{m_1, \dots, m_n\}$, we write $(x_1, \dots, x_n)$ instead of $(\{x_1, \dots, x_n\})$.

$I \triangleleft R$ is finitely generated if $I = (M)$ where M is a finite set. I is a principal ideal, if $I = (a)$ for some $a \in R$.

A domain where every ideal is principal is called a principal ideal domain (PID).

$F[x]$ is a PID if F is a field. A polynomial ring in more variables is not a PID.

Let R be a domain. An element $0 \neq a \in R$ is irreducible if it is not invertible and it cannot be written as a product of non-invertible elements. R is a unique factorization domain (UFD) if:

i) Each $0 \neq a \in R$ can be written in a form $a = up_1 \cdots p_n$, where u is invertible and $p_1, \dots, p_n$ are irreducible.

ii) If $a = vq_1 \cdots q_m$ is another such expression, then $m=n$ and there exists a permutation π of elements $w_1, \dots, w_n$ s.t. $q_i = w_i p_{\pi(i)}$ for each i . We say q_i and $p_{\pi(i)}$ are associated.

Polynomial rings in any number of variables over a field are UFD.

$\text{PID} \Rightarrow \text{UFD}$

$\nLeftarrow$

Proposition: For a ring R the following are equivalent:

- i) Each ideal is finitely generated.
- ii) Each increasing sequence of ideals $I_1 \subseteq I_2 \subseteq I_3 \subseteq \dots$ terminates, i.e. $I_m = I_{m+1} = I_{m+2} = \dots$ for some m .
- iii) Each family of ideals in R has a maximal element (for inclusion).

Proof: Commutative algebra.

Definition: A ring satisfying the above properties is called a noetherian ring (noetherski kolobar).

- 1) Each PID is noetherian.
 $F[x]$ is noetherian if F is a field
- 2) Each quotient R/I of a noetherian ring R is noetherian.

Fact from commutative algebra:

Theorem [Hilbert basis theorem]: If R is noetherian, then $R[x]$ is noetherian.

Remark: If R is noetherian, then the power series ring $R[[x]]$ is noetherian.

Corollary: If F is a field, then $F[x_1, \dots, x_n]$ is noetherian.

Corollary: Every finitely generated algebra over a field is noetherian.

2. Affine varieties and Zariski topology

afine raznoterosti

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We will always work over an algebraically closed field k .

Definition: The n -dimensional affine space over k is

$$A^n = A_k^n = \{(a_1, \dots, a_n) \mid a_i \in k \text{ for each } i\}.$$

As a set, $A^n = k^n$, but k^n has an additional structure of a vector space, so we use a different notation.

affine space - translated vec. space

- not important where the origin is

$n=1$: A^1 : affine line

$n=2$: A^2 : affine plane

Definition: Let $S \subseteq k[x_1, \dots, x_n]$ be a set of polynomials.

The (affine) **zero focus** of S is the set

$$V(S) := \{(a_1, \dots, a_n) \in A^n \mid \forall f \in S, f(a_1, \dots, a_n) = 0\}.$$

This set contains the set of all common zeroes of polynomials in S . If $S = \{f_1, \dots, f_m\}$, then we write $V(f_1, \dots, f_m)$ instead of $V(\{f_1, \dots, f_m\})$.

Definition: A set $X \subseteq A^n$ is an **affine variety** if $X = V(S)$ for some set $S \subseteq k[x_1, \dots, x_n]$.

some authors also additionally assume irreducibility

Examples: i) $\emptyset = V(1)$

ii) $A^n = V(0)$

iii) Each point is an affine variety:

$$\{(a_1, \dots, a_n)\} = V(x_1 - a_1, \dots, x_n - a_n)$$

iv) All affine spaces (and in particular all vector spaces) are affine spaces, as they are solutions of systems of linear equations.

v) Plane affine algebraic curves are affine varieties. They are defined as $\{(a,b) \in A^2 \mid p(a,b) = 0\}$ where $p \in k[x,y]$.
 $= V(p)$

vi) If $p \in k[x_1, \dots, x_n]$, then $V(p) = \{(a_1, \dots, a_n) \in A^n \mid p(a_1, \dots, a_n) = 0\}$ is called a **hypersurface**.

Lemma: If $S_1 \subseteq S_2 \subseteq k[x_1, \dots, x_n]$, then $V(S_2) \subseteq V(S_1)$.

Proof: Obvious.

Proposition: Let $S \subseteq k[x_1, \dots, x_n]$ and I the ideal generated by S . Then $V(I) = V(S)$.

Proof: $S \subseteq I$, so $V(I) \subseteq V(S)$ follows from the Lemma.

($\supseteq$): Assume we have $a \in V(S) \setminus V(I)$. Then there is $f \in I$ such that $f(a) \neq 0$. I is generated by S , so there exist $g_1, \dots, g_m \in k[x_1, \dots, x_n]$ and $h_1, \dots, h_m \in S$ s.t.
 $f = g_1 h_1 + \dots + g_m h_m$.

$$0 \neq f(a) = g_1(a) h_1(a) + \dots + g_m(a) h_m(a) = 0 \quad \text{because } a \in V(S)$$

We get $V(S) = V(I)$. □

Corollary: Varieties in A^n are exactly sets of the form $V(I)$, where $I \triangleleft k[x_1, \dots, x_n]$.

It may happen that $I_1 \neq I_2$, but $V(I_1) = V(I_2)$.

Example: in A^1 : $V(x^2) = \{0\} = V(x)$

$k[x_1, \dots, x_n]$ is noetherian, so all ideals are finitely generated. If $I = (f_1, \dots, f_m)$, then $V(f_1, \dots, f_m) = V(I)$.

Corollary: Affine varieties in A^n are exactly the sets $V(S)$ where S is a finite set.

Lemma: (i) For any family $\{S_j\}_{j \in J}$ of subsets of $k[x_1, \dots, x_n]$ we have $V(\bigcup_{j \in J} S_j) = \bigcap_{j \in J} V(S_j)$.

(ii) For any polynomials $f_1, \dots, f_s, g_1, \dots, g_t \in k[x_1, \dots, x_n]$ we have $V(f_1, \dots, f_s) \cup V(g_1, \dots, g_t) = V(\{f_i, g_j \mid 1 \leq i \leq s, 1 \leq j \leq t\})$.

Proof: $a \in V(\bigcup_{j \in J} S_j) \Leftrightarrow \forall f \in \bigcup_{j \in J} S_j. f(a) = 0$

$$\Leftrightarrow \forall j \in J. \forall f \in S_j. f(a) = 0$$

$$\Leftrightarrow \forall j \in J. a \in V(S_j)$$

$$\Leftrightarrow a \in \bigcap_{j \in J} V(S_j)$$

(ii): Assume $a \in V(f_1, \dots, f_s) \cup V(g_1, \dots, g_t)$. Then

$$a \in V(f_1, \dots, f_s) \text{ or } a \in V(g_1, \dots, g_t).$$

$$\Rightarrow \forall i. f_i(a) = 0 \text{ or } \forall j. g_j(a) = 0$$

In both cases $f_i(a)g_j(a) = 0 \forall i, j$

$$\Rightarrow a \in V(f_i g_j \mid 1 \leq i \leq s, 1 \leq j \leq t).$$

Conversely, assume $a \notin V(f_1, \dots, f_s) \cup V(g_1, \dots, g_t)$.

$$\exists i \text{ s.t. } f_i(a) \neq 0 \text{ and } \exists j \text{ s.t. } g_j(a) \neq 0 \Rightarrow f_i(a) g_j(a) \neq 0$$

$$\Rightarrow a \notin V(f_i g_j \mid 1 \leq i \leq s, 1 \leq j \leq t).$$

◻

Corollary: (1) $\emptyset, \mathbb{A}^n$ are affine varieties

(2) If $\{X_j\}_{j \in J}$ is any family of varieties, then $\bigcap_{j \in J} X_j$ is also an affine variety.

(3) If $X, Y \subseteq \mathbb{A}^n$ are affine varieties, then $X \cup Y$ is an affine variety.

(1)-(3) are axioms of closed sets of some topology, so affine varieties are exactly the closed sets of some topology on $\mathbb{A}^n$. This topology is called the **Zariski topology** on $\mathbb{A}^n$.

On subsets of $\mathbb{A}^n$ we define the Zariski topology as a relative topology: Let $X \subseteq \mathbb{A}^n$ be an arbitrary set. A subset $Z \subseteq X$ is Zariski closed in X if there exists an affine variety $Y \subseteq \mathbb{A}^n$ s.t. $Z = X \cap Y$. In particular, if X is an affine variety, then a set $Z \subseteq X$ is closed $\Leftrightarrow$ it is an affine variety.

If the topology is not mentioned explicitly, we will always mean the Zariski topology.

Examples: Zariski topology on A^1 :

Zariski closed sets are common zeroes of finitely many polynomials. Each nonzero in 1 variable has finitely many zeroes. $\Rightarrow$ All closed sets are finite.

Converse is clear: given a finite set in A^1 , it is easy to find a polynomial whose zeroes are precisely the elements of the given set.

$\Rightarrow$ On A^1 the Zariski topology is equal to the topology of finite complements.

The example shows that the Zariski topology is NOT Hausdorff: every two open sets of A^1 intersect (and the same holds for open subsets in A^n).

Example: Zariski closed sets in A^2 are A^2 , $\emptyset$, finite unions of points and affine algebraic curves.

Open sets in the Zariski topology are complements of varieties.

Definition: Let $p \in K[x_1, \dots, x_n]$. The set $D(p) := A^n \setminus V(p)$ $= \{a \in A^n \mid p(a) \neq 0\}$ is called a distinguished open set (odlikovana odprta množica) of p in A^n . subset
 podmnožica

Example: Distinguished open sets in A^2 are complements of algebraic curves.

$$f, g \in K[x_1, \dots, x_n]$$
$$D(f) \cap D(g) = \{a \in A^n \mid f(a) \neq 0 \wedge g(a) \neq 0\}$$

$$= \{a \in \mathbb{A}^n \mid f(a)g(a) \neq 0\}$$

$$= D(f \cdot g)$$

$\Rightarrow$ The intersection of distinguished open subsets is a distinguished open subset.

Distinguished open subsets form a basis of the Zariski topology: every open subset is a finite union of distinguished open subsets.

Let $U \subseteq \mathbb{A}^n$ be an open subset. Then $Z = \mathbb{A}^n \setminus U$ is closed, so an affine variety. Therefore there exist polynomials $f_1, \dots, f_m$ s.t. $Z = V(f_1, \dots, f_m) = \{a \in \mathbb{A}^n \mid \forall i. f_i(a) = 0\} = \bigcap_{i=1}^m V(f_i) \Rightarrow U = \bigcup_{i=1}^m D(f_i)$.

3. V-I correspondence and Nullstellensatz

"če tega ne prevajajo v angleščino, tudi v slovenščino ne bomo."

Definition: For each subset $X \subseteq \mathbb{A}^n$ we define $I(X) := \{f \in \mathbb{k}[x_1, \dots, x_n] \mid \forall a \in X. f(a) = 0\}$. This is an ideal in $\mathbb{k}[x_1, \dots, x_n]$ called the **ideal of X** . (easy exercise)

$$\{\text{varieties in } \mathbb{A}^n\} \xrightleftharpoons[V]{I} \{\text{ideals in } \mathbb{k}[x_1, \dots, x_n]\}$$

These two maps are not inverse to each other.

For example, we know that $V(x^2) = V(x) \Rightarrow$

$$I(V(x^2)) = I(V(x)).$$

Definition: Let $I_1, I_2 \triangleleft R$. The product of ideals I_1, I_2 is $I_1 I_2 = \left\{ \sum_{i=1}^m a_i b_i \mid m \in \mathbb{N}, a_i \in I_1, b_i \in I_2 \right\}$.

Lemma: Product of ideals is an ideal.

Proof: exercise

Definition: Let $I \triangleleft R$. The **radical** of I is $\sqrt{I} = \text{rad}(I) = \{a \in R \mid a^m \in I \text{ for some } m \in \mathbb{N}\}$. The ideal $I \triangleleft R$ is **radical** if $I = \sqrt{I}$.

Lemma: Radical of an ideal is an ideal.

Exercise: show that if $a^m \in I$ and $b^n \in I$, then $(a+b)^{m+n-1} \in I$.

Example: If $I = ((x-a_1)^{k_1} (x-a_2)^{k_2} \cdots (x-a_r)^{k_r})$, then $\sqrt{I} = ((x-a_1)(x-a_2) \cdots (x-a_r))$. Proof: exercise.

Proposition: (1) $I(\emptyset) = k[x_1, \dots, x_n]$.

(2) $I(A^n) = (0)$

(3) If $I_1 \subseteq I_2$, then $V(I_2) \subseteq V(I_1)$.

(4) If $X_1 \subseteq X_2$, then $I(X_2) \subseteq I(X_1)$.

(5) $X \subseteq V(I(X))$ for each $X \subseteq A^n$.

(6) $S \subseteq I(V(S))$ for each $S \subseteq k[x_1, \dots, x_n]$.

(7) $V(S) = V(I(V(S))) \forall S \subseteq k[x_1, \dots, x_n]$.

(8) If $X \subseteq A^n$ is a variety, then $X = V(I(X))$.

(9) If $X \subseteq A^n$ is any set, then $V(I(X)) = \overline{X}$ (Zariski closure of X), $I(X) = I(\overline{X})$.

(10) $I(X)$ is always a radical ideal.

(11) $I(X) = I(V(I(X))) \forall X \subseteq A^n$.

(12) $V(I) = V(\sqrt{I})$ for each ideal.

(13) $V(I_1) \cup V(I_2) = V(I_1 \cap I_2) = V(I_1 I_2)$ for all ideals I_1, I_2 .

(14) $V(I_1) \cap V(I_2) = V(I_1 + I_2)$ for all ideals I_1, I_2 .

(15) $I(X \cup Y) = I(X) \cap I(Y) \forall$ varieties X, Y

(16) $I(X) + I(Y) \subseteq I(X \cap Y) \forall$ varieties X, Y

Proof: (2) It is clear that 0 vanishes everywhere.
 We have to prove that it is the only such polynomial.
 We prove it with induction on n .

$n=1$: Let f be a polynomial in 1 variable that vanishes everywhere on A^1 . Since k is algebraically closed, it is infinite and the only polynomial that vanishes everywhere it is the zero polynomial.

$n \mapsto n+1$: Let $f \in k[x_1, \dots, x_{n+1}]$ vanish everywhere on A^{n+1} .

Write $f(x_1, \dots, x_n) = \sum_{i=0}^d g_i(x_1, \dots, x_n) x_{n+1}^i$.

Take any $(a_1, \dots, a_n) \in A^n$. Then $f(a_1, \dots, a_n, x_{n+1})$ is a polynomial in 1 variable that vanishes everywhere by the assumption. By case $n=1$, all coefficients of $f(a_1, \dots, a_n, x_{n+1})$ are zero.

$$\Rightarrow \forall i \quad g_i(a_1, \dots, a_n) = 0 \quad \forall (a_1, \dots, a_n) \in A^n$$

$$\Rightarrow g_i \text{ is the zero polynomial } \forall i$$

ind. assumption

$$\Rightarrow f \text{ is the zero polynomial.} \quad \square$$

(11): exercise class

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(7): similar proof

(8): the same as 7

(9): $\bar{X} = V(I(X))$

Clear: $X \subseteq V(I(X))$

Suppose that Z is a variety such that $X \subseteq Z \mid I$

$$I(Z) \subseteq I(X) \quad \setminus \vee$$

$$V(I(Z)) \subseteq V(I(X)) = X$$

$\uparrow$
 Z is a variety

$\Rightarrow V(I(X))$ is the smallest closed set that contains X

$$(13) V(I_1) \cup V(I_2) = V(I_1 \cap I_2) = V(I_1 I_2)$$

Suppose $I_1 = (f_1, \dots, f_s)$ and $I_2 = (g_1, \dots, g_t)$.

Then: $I_1 I_2 = (f_i g_j; 1 \leq i \leq s, 1 \leq j \leq t)$.

$$\begin{aligned} \text{We know that } V(\underbrace{f_1, \dots, f_s}_{I_1}) \cup V(\underbrace{g_1, \dots, g_t}_{I_2}) &= \\ &= V(\underbrace{f_i g_j \mid 1 \leq i \leq s, 1 \leq j \leq t}_{I_1 I_2}) \end{aligned}$$

If $a \in V(I_1) \cup V(I_2)$, then $f(a) = 0 \forall f \in I_1$ or $g(a) = 0 \forall g \in I_2$.

In both cases $h(a) = 0 \forall h \in I_1 \cap I_2 \Rightarrow a \in V(I_1 \cap I_2) \Rightarrow$
 $\Rightarrow V(I_1) \cup V(I_2) \subseteq V(I_1 \cap I_2)$.

$$I_1 I_2 \subseteq I_1 \cap I_2 \Rightarrow V(I_1 \cap I_2) \subseteq V(I_1 I_2)$$

Other parts: exercise. ◻

$V(I(X)) = X$ For a variety X

$I(V(J)) \neq J$ for ideal J
↑
in general

We will use the following result from commutative algebra.

Proposition: Let F be a field and let E be a finitely generated F -algebra which is also a field. Then E is a finite algebraic extension of F .

Corollary: Let A be a finitely generated commutative algebra over k (k alg. closed) and let M be some maximal ideal in A . Then $A/M \cong k$.

Proof: A/M is finitely generated and a field, as M is a maximal ideal. By proposition A/M is a finite algebraic extension of k . k is algebraically closed $\Rightarrow$
 $A/M \cong k$. ◻

Theorem [Weak Nullstellensatz]:

- (1) Maximal ideals in the polynomial ring are exactly the ideals of the form $(x_1 - a_1, \dots, x_n - a_n)$ for some $a_1, \dots, a_n \in k$.
- (2) If J is a proper ideal of $k[x_1, \dots, x_n]$ then $V(J) \neq \emptyset$.

It is crucial to have an algebraically closed field:

over $\mathbb{R}$: $V(x^2 + 1) = \emptyset$

$(x^2 + 1)$ is a maximal ideal in $\mathbb{R}[x]$

Proof: (1) Let $a_1, \dots, a_n \in k$. We want to prove that $M = (x_1 - a_1, \dots, x_n - a_n)$ is a maximal ideal.

Define a ring homomorphism

$$\varphi: k[x_1, \dots, x_n] \longrightarrow k$$

$$x_i \longmapsto a_i$$

$$f(x_1, \dots, x_n) \longmapsto f(a_1, \dots, a_n)$$

φ is surjective: for each $a \in k$ the constant polynomial a maps to a

$$\Rightarrow k \cong k[x_1, \dots, x_n] / \ker \varphi$$

$\hat{\text{field}}$

$\Rightarrow \ker \varphi$ is a maximal ideal

It is enough to show that $\ker \varphi = (x_1 - a_1, \dots, x_n - a_n)$.

Obviously $(x_1 - a_1, \dots, x_n - a_n) \subseteq \ker \varphi$.

For the other inclusion take $f \in \ker \varphi$.

We divide f by $x_1 - a_1$ and the remainder belongs to $k[x_2, \dots, x_n]$. We divide the remainder by $x_2 - a_2$ and get the remainder in $k[x_3, \dots, x_n], \dots$

$$\Rightarrow f = \sum_{i=1}^n g_i(x_i - a_i) + b \quad \text{for some } g_i \in k[x_i, x_{i+1}, \dots, x_n] \\ \text{and } b \in k$$

$$f \in \ker \varphi \Rightarrow$$

$$0 = \varphi(f) = \sum_{i=1}^n \varphi(g_i) \varphi(x_i - a_i) + \varphi(b) = b \Rightarrow b = 0$$

$$\Rightarrow f \in (x_1 - a_1, \dots, x_n - a_n)$$

(We didn't use algebraic closure, $(x_1 - a_1, \dots, x_n - a_n)$ is a maximal ideal in $F[x_1, \dots, x_n]$ for any field F .)

We have to prove there are no other maximal ideals. Let $M \triangleleft k[x_1, \dots, x_n]$ be an arbitrary maximal ideal.

k is alg. closed, k is fin. generated k -algebra so $k[x_1, \dots, x_n]/M \cong k$ by previous corollary.

Define the maps $\pi: k[x_1, \dots, x_n] \xrightarrow{\text{canonical projection}} k[x_1, \dots, x_n]/M = k$
Denote $a_i = \pi(x_i) \in k$ for each i .

π is a ring homomorphism $\Rightarrow \pi(f) = f(\pi(x_1), \dots, \pi(x_n)) = f(a_1, \dots, a_n)$

We already proved $\ker \pi = (x_1 - a_1, \dots, x_n - a_n)$. By the construction $\ker \pi = M$.

(2): If J is a proper ideal in $k[x_1, \dots, x_n]$, it is contained in some maximal ideal M .

By (1), M is of the form $(x_1 - a_1, \dots, x_n - a_n)$ for some $a_1, \dots, a_n \in k$.

$$V(M) = V(x_1 - a_1, \dots, x_n - a_n) = \{(a_1, \dots, a_n)\} \neq \emptyset$$

$$\Rightarrow \emptyset \neq V(M) \subseteq V(J).$$



Corollary: We have mutually inverse bijections

$$\begin{aligned} \{\text{points in } A^n\} & \xrightleftharpoons[V]{I} \{\text{maximal ideals in } k[x_1, \dots, x_n]\} \\ (a_1, \dots, a_n) & \longleftrightarrow (x_1 - a_1, \dots, x_n - a_n) \end{aligned}$$

Theorem [Hilbert's Nullstellensatz]:

$$I(V(J)) = \sqrt{J} \text{ for each } J \triangleleft k[x_1, \dots, x_n].$$

Proof: One inclusion is easy.

If $f \in \sqrt{J}$, then $f^m \in J$ for some m .

If $a \in V(J)$, then $f^m(a) = 0 \Rightarrow f(a) = 0 \Rightarrow f \in I(V(J))$.

($\subseteq$): Let $f \in I(V(J))$.

We consider the ring $k[x_1, \dots, x_n, y]$ (with a variable added) and the ideal

$$\tilde{J} = (J) + (f_y - 1) \triangleleft k[x_1, \dots, x_n, y]$$

one generator is added to J

First we show that $V(\tilde{J})$ is empty.

Suppose $(a_1, \dots, a_n, a_{n+1}) \in V(\tilde{J})$.

$$\Rightarrow \forall g \in \tilde{J}. g(a_1, \dots, a_n, a_{n+1}) = 0$$

If $g \in J \triangleleft k[x_1, \dots, x_n]$, we get $g(a_1, \dots, a_n) = 0$.

$$\Rightarrow (a_1, \dots, a_n) \in V(J)$$

$$f \in I(V(J)) \Rightarrow f(a_1, \dots, a_n) = 0$$

$$(a_1, \dots, a_n, a_{n+1}) \in V(f_y - 1) \Rightarrow f(a_1, \dots, a_n) \cdot a_{n+1} = 1$$

$$\Rightarrow f(a_1, \dots, a_n) \neq 0 \dots \text{contradiction} \Rightarrow V(\tilde{J}) = \emptyset$$

By weak Nullstellensatz we get that $\tilde{J}$ is not proper, so $1 \in \tilde{J}$.

$k[x_1, \dots, x_n]$ is noetherian $\Rightarrow$ there exist $g_1, \dots, g_m \in k[x_1, \dots, x_n]$ such that $J = (g_1, \dots, g_m)$.

$$1 \in (g_1, \dots, g_m, f_y - 1) \Rightarrow \exists p_1, \dots, p_m, q \in k[x_1, \dots, x_n, y].$$

$$1 = p_1 g_1 + \dots + p_m g_m + q (f_y - 1) \quad (*)$$

Let N be the largest number such that y^N appears in each $p_1, \dots, p_m$. We multiply (*) with f^N and rearrange the terms in such a way that each y appears together with f as f_y . We get

$$f^N = P_1(x_1, \dots, x_n, f_y) g_1 + \dots + P_m(x_1, \dots, x_n, f_y) g_m + Q(x_1, \dots, x_n, f_y) \cdot (f_y - 1)$$

We look at this equation modulo $(f_Y - 1)$:

$$F^N \equiv P_1(x_1, \dots, x_n, f_Y)g_1 + \dots + P_m(x_1, \dots, x_n, f_Y)g_m$$

$$\equiv P_1(x_1, \dots, x_n, 1)g_1 + \dots + P_m(x_1, \dots, x_n, 1)g_m \pmod{(f_Y - 1)}$$

$$\Rightarrow F^N - \underbrace{\sum_{i=1}^m P_i(x_1, \dots, x_n, 1)g_i}_{\text{we don't have } y \text{ here}} \in (f_Y - 1) \cap k[x_1, \dots, x_n] = (0)$$

$$\Rightarrow F^N = \sum_{i=1}^m \underbrace{P_i(x_1, \dots, x_n, 1)}_{\in k[x_1, \dots, x_n]} \cdot \underbrace{g_i}_{\in J} \in J \Rightarrow f \in \sqrt{J} \quad \square$$

Corollary: V and J are mutually reverse bijections

$$\{\text{radical ideals}\} \xrightleftharpoons[V]{V} \{\text{affine varieties}\}.$$

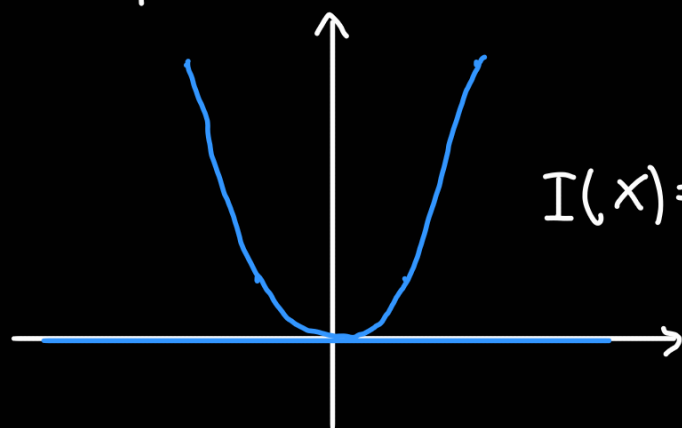
Corollary: $I(X \cap Y) = \sqrt{I(X) + I(Y)}$ if X, Y are affine affine varieties.

Proof: $I(X \cap Y) = I(V(I(X)) \cap V(I(Y)))$

$$= I(V(I(X) + I(Y)))$$

Nullstellensatz $\rightarrow$ $= \sqrt{I(X) + I(Y)}$ □

Example: $X = V(y^2 - x)$, $Y = V(y)$



$$X \cap Y = \{(0,0)\}$$

$$I(X) = I(V(y - x^2)) = \sqrt{(y - x^2)} = (y - x^2)$$

$$I(Y) = (y)$$

↑
principal ideal
generated by
polynomial without
multiple factors

$$I(x) + I(y) = (y - x^2, y) = (y, x^2)$$

This ideal is not radical, as it contains x^2 , but not x
 $\Rightarrow I(x) + I(y) \neq I(x \wedge y)$.

$$I(x \wedge y) = I((0,0)) = (x, y) = \sqrt{(x^2, y)}$$

$I(x) + I(y)$ is not radical, because X and Y have a common tangent in $(0,0)$.

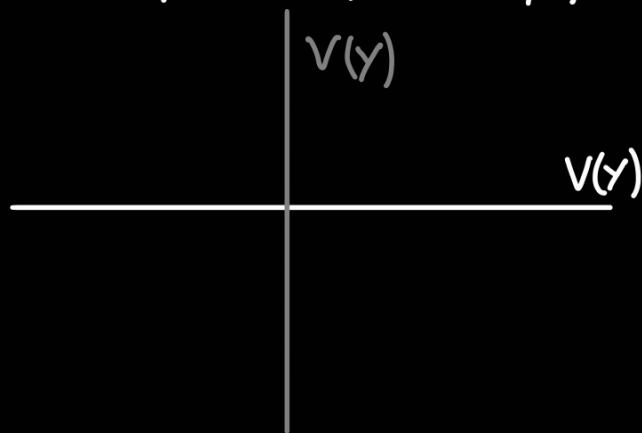
4. Irreducibility of varieties

A topological space X is disconnected if it is a union of two disjoint closed subsets. It is connected otherwise.

Definition: A topological space X is **reducible** if there exist proper closed subsets $X_1, X_2 \subseteq X$ such that $X = X_1 \cup X_2$. X is **irreducible** otherwise.

Definition: An affine variety X is **reducible** if there exist varieties $X_1, X_2 \subsetneq X$ such that $X = X_1 \cup X_2$. X is **irreducible** otherwise.

Example: $V(xy) = V(x) \cup V(y)$ is reducible



Lemma: If X is an irreducible variety, then the following holds:

- 1) Each non-empty open subset of X is dense in X .
(in the Zariski topology)
- 2) Every two non-empty open subsets of X intersect.

Proof: (1) If U is open and $\bar{U} = X$, then we have a decomposition $X = \bar{U} \cup (X \setminus U)$

(2): If U_1, U_2 are open, non-empty, and $U_1 \cap U_2 = \emptyset$, then $X = (X \setminus U_1) \cup (X \setminus U_2)$. $\square$

Definition: Let $I \triangleleft R$. $\swarrow$ slovene: "primeal"

(1) I is a **prime ideal** if $I \neq R$ and the following holds:
If $ab \in I$ for some $a, b \in R$, then $a \in I$ or $b \in I$.

2) I is a **primary ideal** if $I \neq R$ and the following holds: If $ab \in I$ for some $a, b \in R$, then $a \in R$ or $b^m \in I$ for some $m \in \mathbb{N}$. $\swarrow$ slovene: "primarni ideal"

Lemma: A radical of a primary ideal is a prime ideal.

Proof: Let I be primary and $ab \in \sqrt{I}$. Then $(ab)^r = a^r b^r \in I$ for some $r \in \mathbb{N}$.

Since I is primary, we get $a^r \in I$ or $(b^r)^m = b^{rm} \in I$ for some m .
 $\downarrow$ $\downarrow$
 $a \in \sqrt{I}$ $b \in \sqrt{I}$ $\square$

Corollary: A primary ideal which is radical is a prime ideal.

Theorem: $X \subseteq A^n$ is an irreducible variety $\Leftrightarrow I(X)$ is a prime ideal.

Proof: ($\Rightarrow$): Assume X is irreducible and let $fg \in I(X)$. Define $X_1 = X \cap V(f)$ and $X_2 = X \cap V(g)$. Then $X_1, X_2 \subseteq X \Rightarrow X_1 \cup X_2 \subseteq X$.

Let $a \in X$. $fg \in I(X) \Rightarrow f(a)g(a) = 0 \Rightarrow f(a) = 0$ or $g(a) = 0 \Rightarrow a \in V(f)$ or $a \in V(g) = X = X_1 \cup X_2$.

X_1, X_2 are closed, so by irreducibility of X one of them is equal to X .

WLOG: $X_1 = X$ ($X_1 = X \cap V(f)$)

$\Rightarrow X \subseteq V(f) \Rightarrow \forall a \in X. f(a) = 0 \Rightarrow f \in I(X)$

($\Leftarrow$): Assume $I(X)$ is a prime ideal. Let $X = X_1 \cup X_2$ for some $X_1, X_2 \subseteq X$ varieties and suppose that $X_1 \neq X$. We will show that $I(X) = I(X_2)$.

$X_2 \subseteq X \Rightarrow I(X) \subseteq I(X_2)$

For the other inclusion take $g \in I(X_2)$.

$X_1 \subsetneq X \Rightarrow I(X) \subsetneq I(X_1) \Rightarrow \exists f \in I(X_1) \setminus I(X)$

Let $a \in X$ be arbitrary. Then $a \in X_1$ or $a \in X_2$, so $f(a) = 0$ or $g(a) = 0$. So we have $f(a)g(a) = 0$ for each $a \in X$. $\Rightarrow fg \in I(X)$.

$I(X)$ is prime and $f \notin I(X)$, so $g \in I(X)$

$\Rightarrow I(X_2) \subseteq I(X) \Rightarrow I(X_2) = I(X)$

$\Rightarrow V(I(X_2)) = V(I(X)) \Rightarrow X$ is irreducible.



Corollary: A hypersurface $V(f)$ where f is a square-free polynomial is irreducible $\Leftrightarrow f$ is irreducible. $\uparrow$

$$f = g_1^{h_1} \cdots g_r^{h_r}, \quad g_i \text{ irreducible} \Rightarrow h_i = 1 \quad \forall i$$

Corollary: $\mathbb{A}^n$ is irreducible

Proof: $I(\mathbb{A}^n) = (0)$ is a prime ideal. $\square$

Corollary: V and I are mutually inverse bijections

$$\left\{ \begin{array}{c} \text{irreducible} \\ \text{varieties} \end{array} \right\} \xrightleftharpoons[V]{I} \left\{ \text{prime ideals} \right\}.$$

Remark:

algebra	geometry
$k[x_1, \dots, x_n]$	$\mathbb{A}^n$
maximal ideals	points
radical ideals	affine varieties
prime ideals	irreducible affine varieties

$$\begin{array}{c} \xrightarrow{V} \\ \xleftarrow{I} \end{array}$$

Theorem: Each affine variety X can be decomposed as a union $X = X_1 \cup \cdots \cup X_m$ where $m \in \mathbb{N}_0$ and $X_1, \dots, X_m$ are non-empty irreducible varieties.

Moreover, if $X_i \subseteq X_j$ whenever $i \neq j$, then the decomposition is unique up to permutation.

Definition: If $X = X_1 \cup \cdots \cup X_m$ where $X_1, \dots, X_m$ are irreducible varieties and $X_i \not\subseteq X_j$, whenever $i \neq j$, then $X_1, \dots, X_m$ are called **irreducible components** of X .

shvene: nerazcepne komponente

Proof: If $X \neq \emptyset$, then the decomposition exists (for $m=n$) and it is unique.

Assume now that $X \neq \emptyset$.

Existence of the composition:

Assume that no decomposition $X = X_1 \cup \dots \cup X_m$ where $X_1, \dots, X_m$ are irreducible, exists. Then X is reducible (as otherwise $X = X$ is such decomposition for $m=1$).

$X = X_1 \cup X_1'$ for some varieties X_1 and X_1' , and at least one of them is not a union of irr. varieties.

WLOG: this is X_1 , X_1 has to be reducible: $X_1 = X_2 \cup X_2'$ for some var. X_2, X_2' and at least one of them is not a union of irr. varieties.

$\vdots$

We get a strictly decreasing chain of varieties:

$$X \supsetneq X_1 \supsetneq X_2 \supsetneq \dots \quad | \quad I$$

$$I(X) \subsetneq I(X_1) \subsetneq I(X_2) \subsetneq \dots$$

This is a strictly increasing sequence of ideals in $k[x_1, \dots, x_n]$, which contradicts the noetherian property.
 $\Rightarrow X$ can be decomposed into a union of irreducible varieties.

Uniqueness: $X = X_1 \cup \dots \cup X_r = X_1' \cup \dots \cup X_s'$ where X_i, X_i' are irreducible, $X_i \not\subseteq X_j, X_i' \not\subseteq X_j'$ whenever $i \neq j$.
Take arbitrary $i \in \{1, \dots, r\}$.

$$\begin{aligned} X_i &= X_i \cap X = X_i \cap (X_1' \cup \dots \cup X_s') \\ &= \bigcup_{j=1}^s (X_i \cap X_j'), \quad X_i \text{ is irreducible} \end{aligned}$$

therefore $X_i = X_i \cap X_j'$ for some $j \Rightarrow X_i \subseteq X_j'$

The same argument shows that there exists $l \in \{1, \dots, r\}$ such that $X_j' \subseteq X_l \Rightarrow X_i \subseteq X_j' \subseteq X_l$.

By assumption $l=i \Rightarrow X'_j = X_i$.

$\Rightarrow$ We get uniqueness (and in particular, $r=s$). ▣

Remark: The crucial part was to show the fact that there does not exist an infinite sequence of closed subsets, each properly contained in the previous one, $X \supsetneq X_1 \supsetneq X_2 \supsetneq \dots$.

We say that varieties are **noetherian topological spaces**.

Remark: In commutative algebra an important theorem says that each ideal in a noetherian ring can be written as an intersection of primary ideals. Using this fact we could prove the theorem as follows:

$$I(X) = Q_1 \cap Q_2 \cap \dots \cap Q_m, \quad Q_i \text{ primary} / V$$

$$X = V(I(X)) = V(Q_1 \cap \dots \cap Q_m)$$

$$= V(Q_1) \cup \dots \cup V(Q_m)$$

To show that $V(Q_i)$ are irreducible, apply I :

$$I(V(Q_i)) = \sqrt{\underbrace{Q_i}_{\text{primary}}} = \text{prime ideal} \Rightarrow V(Q_i) \text{ irreducible.}$$

If we use only minimal prime ideals over $I(X)$ then we also get the uniqueness statement.

17. oktober 2025

II PROJECTIVE VARIETIES

1. Projective space and projective varieties

Let V be a finite-dimensional vector space over $\mathbb{k}$ (still alg. closed). On $V \setminus \{0\}$ we define a relation $u \sim v \Leftrightarrow \exists \lambda \in \mathbb{k} \setminus \{0\}$ such that $v = \lambda u$. This is an equivalence relation. The quotient set $V/\sim$ is denoted by $\mathbb{P}V$ and called the projective space associated to V . Its elements are equivalence classes in V , so lines through the origin. Dimension of $\mathbb{P}V := \dim V - 1$.

The most common situation is when $V = \mathbb{k}^{n+1}$ for some $n \in \mathbb{N}$. In this case $\dim V = n+1$, so $\dim \mathbb{P}V = n$. Instead of $\mathbb{P}\mathbb{k}^{n+1}$ we write $\mathbb{P}_{\mathbb{k}}^n$ or more usually $\mathbb{P}^n$. We call $\mathbb{P}^n$ the n -dimensional projective space. Its elements (which are lines through 0) are called projective points.

- $\mathbb{P}^1$ is projective line
- $\mathbb{P}^2$ is projective plane

When we work in $\mathbb{P}^n$, we index coordinates in $\mathbb{k}^{n+1}$ from 0 to n : $(x_0, x_1, \dots, x_n)$.

The equivalence class of the point $(x_0, x_1, \dots, x_n)$ is denoted by $(x_0 : x_1 : \dots : x_n)$. In literature there is also notation $[x_0 : x_1 : \dots : x_n]$ or $[x_0, x_1, \dots, x_n]$.

So $(x_0 : x_1 : \dots : x_n)$ is the line in $\mathbb{k}^{n+1}$ through $(x_0, x_1, \dots, x_n)$ and the origin; $x_0, x_1, \dots, x_n$ are called homogeneous coordinates of the point $(x_0 : x_1 : \dots : x_n) \in \mathbb{P}^n$. $x_0, \dots, x_n$ are not all zero.

The points $(x_0 : x_1 : \dots : x_n)$ and $(y_0 : y_1 : \dots : y_n)$ are equal $\Leftrightarrow \exists \lambda \in \mathbb{K} \setminus \{0\}$ s.t. $y_i = \lambda x_i \quad \forall i = 0, 1, \dots, n$.

We can embed $\mathbb{A}^n$ into $\mathbb{P}^n$. For each $i = 0, 1, \dots, n$, we define $U_i = \{(x_0 : x_1 : \dots : x_n) \in \mathbb{P}^n ; x_i \neq 0\}$.

If $(x_0 : x_1 : \dots : x_n) = (y_0 : y_1 : \dots : y_n)$, then $y_j = \lambda x_j \quad \forall j = 0, \dots, n$ and $\lambda \neq 0$, so $x_i \neq 0 \Leftrightarrow y_i \neq 0 \Rightarrow$ the sets U_i are well defined.

We define a map: $\mathbb{A}^n \longrightarrow U_i$

$$(x_1, \dots, x_n) \longmapsto (x_1 : \dots : x_i : 1 : x_{i+1} : \dots : x_n).$$

This is a bijection with the inverse

$$U_i \longrightarrow \mathbb{A}^n$$

$$(x_0 : x_1 : \dots : x_n) \longmapsto \left(\frac{x_0}{x_i}, \frac{x_1}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_n}{x_i} \right).$$

This map is well defined, because $\frac{\lambda x_i}{\lambda x_i} = \frac{x_i}{x_i}$.

$\Rightarrow$ We can identify $\mathbb{A}^n$ with U_i , (most commonly with U_0) and consider it as a subspace of $\mathbb{P}^n$.

$U_0, U_1, \dots, U_n$ are usually called **affine charts** of $\mathbb{P}^n$.

$\mathbb{P}^n \setminus U_i$ consists of all points $(x_0 : x_1 : \dots : x_n)$ with $x_i = 0$.

But $x_0, x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n$ are not all zero, so we have

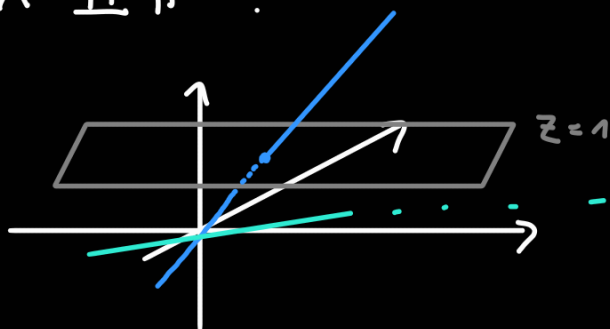
$$\mathbb{P}^n \setminus U_i \longrightarrow \mathbb{P}^{n-1}$$

$$(x_0 : \dots : x_{i-1} : 0 : x_{i+1} : \dots : x_n) \longmapsto (x_0 : \dots : x_{i-1} : x_{i+1} : \dots : x_n)$$

This is a bijection. $\swarrow$ disjoint union

We have $\mathbb{P}^n = \mathbb{A}^n \sqcup \mathbb{P}^{n-1}$.

Example: $\mathbb{P}^2$



Lines in xy -plane are in bijection with points in $\mathbb{P}^1$.

$A^n \perp \mathbb{P}^{n-1}$ → This is usually called the **hyperplane at infinity**.

We want to study zero loci of polynomials in $\mathbb{P}^n$.
In $\mathbb{P}^n$ we have homogeneous coordinates:

$$(x_0 : x_1 : \dots : x_n) = (\lambda x_0 : \lambda x_1 : \dots : \lambda x_n) \text{ for } \lambda \neq 0$$

so $f(x_0, x_1, \dots, x_n)$ is not well defined. We restrict to homogeneous polynomials.

Definition: A polynomial $f \in k[x_0, x_1, \dots, x_n]$ is **homogeneous of degree d** if

$$f(\lambda x_0, \lambda x_1, \dots, \lambda x_n) = \lambda^d f(x_0, x_1, \dots, x_n)$$

For each $\lambda \in k \setminus \{0\}$.

Since k is infinite, this is equivalent to that all monomials of f are of degree d .

Example: $x_0^2 x_1 + x_2^3 - x_4 x_5 x_6$ is homogeneous of degree 3
 $x_0^3 x_1 + x_2^2$ is not homogeneous
 $\uparrow$ $\uparrow$
 deg 3 deg 2

Definition: Let $S \subseteq k[x_0, x_1, \dots, x_n]$ be a set of homogeneous polynomials. The set

$$V(S) := \{(x_0 : x_1 : \dots : x_n) \in \mathbb{P}^n \mid f(x_0, \dots, x_n) = 0 \text{ for all } f \in S\}$$

is called the **projective zero locus** of S . A set

$X \subseteq \mathbb{P}^n$ is a **projective variety** if $X = V(S)$ for some set S of homogeneous polynomials.

If $S = \{f_1, \dots, f_m\}$ we write $V(f_1, \dots, f_m)$ instead of $V(\{f_1, \dots, f_m\})$.

Projective zero loci are well defined:

If $F \in S$ is homogeneous of degree d , then

$$F(\lambda x_0, \lambda x_1, \dots, \lambda x_n) = \lambda^d F(x_0, \dots, x_n) \text{ for all } \lambda \in k \setminus \{0\}.$$

so we get $F(\lambda x_0, \dots, \lambda x_n) = 0 \Leftrightarrow F(x_0, \dots, x_n) = 0$.

Remark: $V(S)$ can mean affine zero locus or projective zero locus. When there can be confusion, we will write $V_a(S)$ or $V_p(S)$.

Examples of projective varieties

(1) $\mathbb{P}^n = V(0)$

(2) $\emptyset = V(1)$, but also $\emptyset = V(x_0, x_1, \dots, x_n)$.

(3) If V is a vector subspace of k^{n+1} , then $\mathbb{P}V$ is a projective variety in $\mathbb{P}^n$, because vector subspaces are defined by homogeneous linear equations. $\mathbb{P}V$ is called the **linear subspace** of $\mathbb{P}^n$.

(4) Each point is a projective variety: If $a = (a_0 : a_1 : \dots : a_n)$, then $V(a_i x_j - a_j x_i \mid 0 \leq i, j \leq n) = \{a\}$. Proof: HW

October 21, 2025

We may define varieties also in product of affine and projective spaces. For example, a variety in $A^m \times \mathbb{P}^n$ is a zero locus of a set of polynomials in $k[x_1, \dots, x_m, y_0, \dots, y_n]$ that are homogeneous in the variables $y_0, \dots, y_n$.

Example: $V(x_1^2 y_0^3 - x_2 y_1 y_2^2) \subseteq A^2 \times \mathbb{P}^2$.

A variety in $\mathbb{P}^m \times \mathbb{P}^n$ is a zero locus of a set of polynomials in $k[x_0, x_1, \dots, x_m, y_0, \dots, y_n]$ that are homogeneous in $x_0, \dots, x_m$ and (maybe of a different degree) in $y_0, \dots, y_n$.

Example: $V(x_0^3 x_1 y_2^2 - x_2^3 y_0 y_1) \subseteq \mathbb{P}^2 \times \mathbb{P}^2$

$A^m \times \mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_s}$ similarly

2. Connection between affine and projective varieties

Definition: (1) An affine variety $X \subseteq \mathbb{A}^{n+1}$ is a **cone** (stožec) if $0 \in X$ and for each $a \in X$ and each $\lambda \in \mathbb{k}$ we have $\lambda a \in X$.

(2) If $X \subseteq \mathbb{A}^{n+1}$ is a cone then the **projectivization** of X is defined by

$$\mathbb{P}X := \{ (a_0 : \dots : a_n) \in \mathbb{P}^n \mid (a_0, a_1, \dots, a_n) \in X \setminus \{0\} \} \subseteq \mathbb{P}^n.$$

(3) If $X \subseteq \mathbb{P}^n$ is a projective variety, then the **cone over X** is defined by $\mathcal{C}(X) := \{0\} \cup \{ (a_0, \dots, a_n) \in \mathbb{A}^{n+1} \mid (a_0 : \dots : a_n) \in X \} \subseteq \mathbb{A}^{n+1}$.

If X is a cone and $(a_0, a_1, \dots, a_n) \in X$ and $\lambda \in \mathbb{k}$. Since X is a cone, we have $(\lambda a_0, \lambda a_1, \dots, \lambda a_n) \in X \Rightarrow \mathbb{P}X$ is well-defined.

Proposition: If $X \subseteq \mathbb{A}^{n+1}$ is a cone, then $\mathbb{P}X$ is a projective variety.

Proof: X is an affine variety, so $X = V_{\mathbb{A}}(S)$ for some $S \subseteq \mathbb{k}[x_0, \dots, x_n]$. We can assume that $S = I_{\mathbb{A}}(X)$. Let $f \in I_{\mathbb{A}}(X)$ is arbitrary. Write $f = \sum_{i=0}^d f_i$, where each f_i is homogeneous of degree i . Take arbitrary $a \in X$ and arbitrary $\lambda \in \mathbb{k}$. Since X is a cone, we have $\lambda a \in X$, so $f(\lambda a) = 0$.

$$0 = f(\lambda a) = \sum_{i=0}^d f_i(\lambda a) = \sum_{i=0}^d \lambda^i f_i(a)$$

This holds for each $\lambda \in \mathbb{k}$, $\mathbb{k}$ infinite $\Rightarrow f_i(a) = 0 \ \forall i$

This holds for each $a \in X$, so $f_i \in I_{\mathbb{A}}(X)$ for each i .

We showed that $I_{\mathbb{A}}(X)$ can be generated by homogeneous

polynomials. Let $I_a(X) = (S')$ where S' is a set of homogeneous polynomials. Then

$$\begin{aligned} PX &= \left\{ (a_0: \dots: a_n) \in \mathbb{P}^n \mid (a_0, \dots, a_n) \in \overset{V(S')}{X} \setminus \{0\} \right\} \\ &= \left\{ (a_0: \dots: a_n) \in \mathbb{P}^n \mid f(a_0, \dots, a_n) = 0 \ \forall f \in S' \right\} \end{aligned}$$

$$= V_p(S')$$

$\uparrow$
 S' is a set of homogeneous polynomials

□

We proved two more things:

Corollary: If S is a set of homogeneous polynomials then $IPV_a(S) = V_p(S)$.

Corollary: X cone $\Rightarrow I_a(X)$ generated by hom. polynomials.

Proposition: The cone over a projective variety is a cone.

Proof: If $X \neq \emptyset$, then $C(X) = \{0\}$ which is a cone.

Assume that $X \subset \mathbb{P}^n$ is a non-empty projective variety. Then $X = V_p(S)$ for some set S of non-constant polynomials. If $f \in S$ is hom. of degree d , then $f(\lambda a) = \lambda^d f(a) \ \forall \lambda \in k$ and $\forall a \in X \Rightarrow f(0) = 0$.

$$\begin{aligned} C(X) &= \{0\} \cup \{(a_0, \dots, a_n) \in A^{n+1} \mid (a_0: \dots: a_n) \in X\} \\ &= \{(a_0, a_1, \dots, a_n) \in A^{n+1} \mid f(a_0, a_1, \dots, a_n) = 0 \ \forall f \in S\} \\ &= V_a(S) \end{aligned}$$

We know $0 \in C(X)$ and that if $a \in V_a(S)$, $\lambda \in k$, then $f(\lambda a) = \lambda^d f(a) = 0$, so $\lambda a \in V_a(S)$. □

We proved also:

Corollary: If S is a set of homogeneous polynomials then $CV_p(X) = V_a(S)$.

Corollary: The maps

$$\begin{array}{ccc} X & \xrightarrow{\quad} & \mathbb{P}X \\ C(X) & \xleftarrow{\quad} & X \end{array}$$

give bijective correspondence between the cones in A^n and projective varieties in $\mathbb{P}^n$.

Proof: Both corollaries tell that $\mathbb{P}X$ is a projective variety if X is a cone and that $C(X)$ is a cone if X is a projective variety. We have to prove bijectivity.

Suppose $X \subseteq A^{n+1}$ is a cone. Then we know from one of the corollaries above that $X = V_a(S)$ for a set of hom. poly. $S \subseteq k[x_0, \dots, x_n]$. By the other two corollaries:
 $C(\mathbb{P}X) = C(\mathbb{P}V_a(S)) = C(V_p(S)) = V_a(S) = X$.

Similarly $\mathbb{P}(C(X)) = X$ if X is a projective variety. $\square$

Corollary: Each projective variety is a zero locus of a finite set of homogeneous polynomials.

Proof: $X = V_p(S)$ for some set S of hom. polynomials. Then $C(X) = V_a(X)$. Let $J = I_a(C(X))$. Then we know that J is finitely generated ($k[x_0, \dots, x_n]$ noetherian) $J = (f_1, \dots, f_n)$. We proved before that homogeneous parts lie in J , and they obviously generate J . So we have a finite set S' of hom. polynomials

that generate S .

$$\Rightarrow X = \mathbb{P}(C(X)) = \mathbb{P}(V_a(S)) = \mathbb{P}(V_a(S')) = V_p(S') \quad \square$$

As in the affine case, we can use this corollary to prove:

Lemma: (1) If $\{S_j\}_{j \in J} \subseteq K[x_0, \dots, x_n]$ is a family of sets of homogeneous polynomials, then

$$V_p(\bigcup_{j \in J} S_j) = \bigcap_{j \in J} V_p(S_j).$$

(2) If $f_1, \dots, f_s, g_1, \dots, g_t \in K[x_0, \dots, x_n]$ are homogeneous polynomials, then

$$V_p(f_1, \dots, f_s) \cup V_p(g_1, \dots, g_t) = V_p(f_i g_j; \substack{1 \leq i \leq s \\ 1 \leq j \leq t}).$$

Corollary: (1) $\emptyset$ and $\mathbb{P}^n$ are projective varieties.

(2) The intersection of any family of projective varieties is a projective variety.

(3) The union of finitely many projective varieties is a projective variety.

Projective varieties are therefore exactly the closed sets on some topology on $\mathbb{P}^n$ - **Zariski topology on $\mathbb{P}^n$** .

As in the affine case, the Zariski topology on subsets of $\mathbb{P}^n$ is the relative topology:

Let $X \subseteq \mathbb{P}^n$. A set $Z \subseteq X$ is Zariski-closed if there exists a proj. var. $Y \subseteq \mathbb{P}^n$ s.t. $Z = X \cap Y$. If X is a proj. var., then its closed subsets are exactly the subvarieties.

As in the affine case, we also define distinguished open subsets: $D(f) = \mathbb{P}^n \setminus V_p(f)$ where f is a hom. poly.

In a similar way we can define the Zariski topology in any product $A^n \times \mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_s}$.

Definition: Let $X \subseteq \mathbb{P}^n$. X is **reducible** if $X = X_1 \cup X_2$ for some proper closed subsets $X_1, X_2 \subsetneq X$, **irreducible** otherwise.

If X is a proj. var., then X is reducible $\Leftrightarrow X$ is a union of two proper subvarieties.

Theorem: Each proj. var. $X \subseteq \mathbb{P}^n$ can be written as $X = X_1 \cup X_2 \cup \dots \cup X_m$ where $m \in \mathbb{N}$ and $X_1, \dots, X_m$ are irreducible proj. var. Moreover, if $X_i \subsetneq X_j$ whenever $i \neq j$, then this decomposition is unique up to an order. In this case $X_1, \dots, X_m$ are called **irreducible components** of X .

$$\mathbb{P}^n = A^n \sqcup \mathbb{P}^{n-1}$$

Recall: $U_i = \{(x_0: \dots: x_n) \in \mathbb{P}^n; x_i \neq 0\} = D(x_i)$

We identified U_i with A^n .

$U_i = D(x_i)$ are open in Zariski topology, moreover, $\{U_0, \dots, U_n\}$ is an open cover of $\mathbb{P}^n$.

$\Rightarrow$ We can consider A^n as an open subset of $\mathbb{P}^n$.

We have 2 Zariski topologies on A^n : one defined by affine varieties and one as a relative topology in $\mathbb{P}^n$.

Are they equal?

We will identify A^n with U_0 .

Definition: Let $f \in k[x_0, \dots, x_n]$ be a homogeneous polynomial.

Dehomogenization of f is the polynomial

$$f^{(d)} := f(1, x_1, \dots, x_n) \in k[x_1, \dots, x_n].$$

Dehomogenization is evaluation $x_0=1$, so it is a ring homomorphism, so:

$$(fg)^{(d)} = f^{(d)} \cdot g^{(d)},$$

$$(f+g)^{(d)} = f^{(d)} + g^{(d)}.$$

Definition: Let $f \in k[x_1, \dots, x_n]$ be a non-zero polynomial of degree d . Then

$$f^{(n)} := x_0^d f\left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right)$$

is a polynomial, called the homogenization of f .

Example: $f(x_1, x_2, x_3) = x_1^3 - x_2 x_3 + 2x_2^4$

$$\deg f = 4 \Rightarrow f^{(4)}(x_0, x_1, x_2, x_3) = x_0 x_1^3 - x_0^2 x_2 x_3 + 2x_2^4$$

We have $(fg)^{(n)} = f^{(n)} g^{(n)}$, but $(f+g)^{(n)} \neq \underbrace{f^{(n)} + g^{(n)}}_{\text{this polynomial may not be homogeneous}}$

Proposition: Each affine variety $X \subseteq A^n \cong U_0 \subseteq \mathbb{P}^n$ is of the form $X = Z \cap U_0$ for some projective variety Z . More precisely, if $X = V_a(f_1, \dots, f_m)$, we may take $Z = V_p(f_1^{(n)}, \dots, f_m^{(n)})$.

Proof: Let f_i be of degree d_i for each i .

$$Z \cap U_0 = V_p(f_1^{(n)}, \dots, f_m^{(n)}) \cap U_0$$

$$\begin{aligned}
&= \{ (a_0 : a_1 : \dots : a_n) \in \mathbb{P}^n \mid a_0 \neq 0, \forall i: f_i^{(n)}(a_0, \dots, a_n) = 0 \} \\
&= \{ (1 : \frac{a_1}{a_0} : \dots : \frac{a_n}{a_0}) \in \mathbb{P}^n \mid a_0 \neq 0, \forall i: f_i^{(n)}(1, \frac{a_1}{a_0}, \dots, \frac{a_n}{a_0}) = 0 \} \\
&= \{ (1 : \frac{a_1}{a_0} : \dots : \frac{a_n}{a_0}) \in \mathbb{P}^n \mid a_0 \neq 0, \forall i: d_i \cdot f_i(1, \frac{a_1}{a_0}, \dots, \frac{a_n}{a_0}) = 0 \} \\
&= \{ (1 : \frac{a_1}{a_0} : \dots : \frac{a_n}{a_0}) \in \mathbb{P}^n \mid a_0 \neq 0, \forall i: f_i(1, \frac{a_1}{a_0}, \dots, \frac{a_n}{a_0}) = 0 \} \\
&\cong \{ (b_1, \dots, b_n) \in A^n \mid \forall i: f_i(b_1, \dots, b_n) = 0 \} \\
&= V_A(f_1, \dots, f_n) = X
\end{aligned}$$

Corollary: Both Zariski topologies on A^n coincide.

We often study open subsets of projective varieties. Such sets are called **quasiprojective** varieties. Important examples of quasiprojective varieties are proj. varieties and affine varieties.

Definition: Let $X \subseteq A^n \subseteq \mathbb{P}^n$ be an affine variety. The **projective closure** of X is the smallest projective variety that contains X . Notation: $\bar{X}$

In general $\bar{X} \neq V_P(f_1^{(n)}, \dots, f_m^{(n)})$ if $X = V_A(f_1, \dots, f_m)$.

Example: $X = V_A(x_1, x_2 - x_1^2) = \{(0,0)\}$. $\bar{X}$ also has to be one point: $\bar{X} = \{[1:0:0]\}$, but $V_P(x_1, x_0 x_2 - x_1^2) = \{[1:0:0], [0:0:1]\} = \bar{X}$.

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Proposition: Let $X \subseteq A^n = U_0 \subseteq \mathbb{P}^n$ be an affine variety and $\bar{X} \subseteq \mathbb{P}^n$ its projective closure. Then:

- (1) $\bar{X} \cap U_0 = X$ (2) X irreducible $\Rightarrow \bar{X}$ irreducible
- (3) No irreducible component of $\bar{X}$ lies in $V_P(x_0)$ (=hyperplane at infinity).

Proof: (1) We know that $X = Z \cap U_0$ for some projective variety Z . Z is a projective variety that contains X , so it also contains $\bar{X}$.

(2) Suppose that $\bar{X} = Z_1 \cup Z_2$ for some projective varieties Z_1, Z_2 . By (1) we get

$$X = \bar{X} \cap U_0 = (Z_1 \cap U_0) \cup (Z_2 \cap U_0)$$

$Z_1 \cap U_0$ and $Z_2 \cap U_0$ are affine varieties, so irreducibility of X implies $X = Z_i \cap U_0$ for some $i=1,2$. Z_i is a projective variety that contains X , so $\bar{X} \subseteq Z_i \Rightarrow \bar{X}$ irreducible.

(3) Let $\bar{X} = Z_1 \cup \dots \cup Z_m$ be the decomposition into irreducible components. Suppose that $Z_1 \subseteq V_p(x_0)$.

$$\begin{aligned} X &= \bar{X} \cap U_0 = (Z_1 \cup \dots \cup Z_m) \cap U_0 \\ &= (Z_1 \cap U_0) \cup ((Z_2 \cup \dots \cup Z_m) \cap U_0) \\ &\quad \parallel \\ &= (Z_2 \cup \dots \cup Z_m) \cap U_0 \end{aligned}$$

$Z_2 \cup \dots \cup Z_m$ is a projective variety that contains X , so it contains $\bar{X}$ and therefore

$$Z_1 \subseteq Z_2 \subseteq \dots \subseteq Z_m$$

Z_1 is irreducible, therefore $Z_1 \subseteq Z_i$ for some $i=2, \dots, m$.

This contradicts the fact that Z_i are components. $\square$

3. Projective algebra-geometry correspondence

Definition: A ring/algebra R is **graded** if we can write it as a direct sum of abelian groups/ k -vector spaces $R = \bigoplus_{d=0}^{\infty} R_d$ such that $R_d \cdot R_e \subseteq R_{d+e}$ for all $d, e \in \mathbb{N}_0$.

If $f \in R_d$ and $g \in R_e$, then $fg \in R_{d+e}$.

slovene: stopničast
kolobar

We say that elements of $R_d \setminus \{0\}$ are **homogeneous of degree d** .

Example: $R = k[x_0, x_1, \dots, x_n]$ is a graded k -algebra:

$R = \bigoplus_{d=0}^{\infty} R_d$, where $R_d = \{0\} \cup \{\text{homogeneous polynomials of degree } d\}$.

Let $f \in R$. Then f can be uniquely decomposed as $f = \sum_{d=0}^{\infty} f_d$ where $f_d \in R_d$ for each d and only finitely many f_d 's are nonzero. The decomposition $f = \sum_{d=0}^{\infty} f_d$ is called the **homogeneous decomposition** of f .

If $f \neq 0$, then the **degree** of f is the largest d s.t. $f_d \neq 0$.

Definition: Let R be a graded ring/algebra. An ideal $I \triangleleft R$ is **homogeneous** if it can be generated by homogeneous elements.

Example: $I = (x_1, x_2 - x_1^2) = I(x_1, x_2)$ is a homogeneous ideal.

Example: If X is a cone, then we showed that $I_a(X)$ is homogeneous.

Proposition: Let $R = \bigoplus_{d=0}^{\infty} R_d$ be a graded ring and $J, J_1, J_2 \triangleleft R$.

Then the following holds:

(1) J is homogeneous $\Leftrightarrow$ for each $f \in J$ with homogeneous decomposition $f = \sum_{d=0}^{\infty} f_d$ we have $f_d \in J$ for each d .

(2) J, J_1, J_2 homogeneous $\Rightarrow J_1 + J_2, J_1 \cap J_2, J_1 J_2, \sqrt{J}$ homogeneous

(3) J homogeneous, then R/J is a graded ring with the homogeneous decomposition:

$$R/J = \bigoplus_{d=0}^{\infty} R_d / (R_d \cap J).$$

|| is isomorphism theorem
 $(R_d + J) / J$

(4) If R is noetherian and J is a homogeneous ideal, then J can be generated by finitely many homogeneous elements.

Proof: Exercise.

Definition: (1) For a homogeneous ideal $J \triangleleft k[x_0, x_1, \dots, x_n]$ we define: $V(J) = V_p(J) := \{x \in \mathbb{P}^n \mid f(x) = 0 \ \forall \text{ homogeneous } f \in J\}$.

(2) For $X \subseteq \mathbb{P}^n$ we define the **ideal of X** as

$$I(X) = I_p(X) := \{f \in k[x_0, \dots, x_n] \text{ homogeneous} \mid f(x) = 0 \ \forall x \in X\}.$$

Remark: (1) $V(J)$ is well defined, because it is defined only using homogeneous polynomials. If S is a homogeneous set of generators of J , then $V_p(J) = V_p(S)$. $V_p(J)$ is a projective variety.

(2) The set of all homogeneous polynomials vanishing on X is not an ideal. To get $I_p(X)$, we must take the ideal generated by them.

Lemma: (1) If $X \subseteq Y \subseteq \mathbb{P}^n$, then $I_p(Y) \subseteq I_p(X)$.

(2) For homogeneous ideals $I_1 \subseteq I_2$ we have $V_p(I_2) \subseteq V_p(I_1)$.

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Proposition: Let $X \subseteq \mathbb{P}^n$ be a non-empty projective variety. Then $I_p(X) = I_a(C(X))$.

Proof: ($\subseteq$): Let $f \in I_p(X)$ and $f = \sum_{d=0}^{\infty} f_d$ be the homogeneous decomposition of f .

We know that $I_p(X)$ is a homogeneous ideal, so $f_d \in I_p(X)$ for each d .

$$\Rightarrow f_d(x_0, x_1, \dots, x_n) = 0 \quad \forall (x_0, x_1, \dots, x_n) \in X$$

$\Rightarrow f_d(x_0, \dots, x_n) = 0$ for each point in the cone

$$\Rightarrow f_d \in I_a(C(X)) \text{ for each } d \Rightarrow f \in I_a(C(X)).$$

($\supseteq$): We proved that $I_a(C(X))$ is a homogeneous ideal, so it is generated by homogeneous elements.

$$\Rightarrow \text{Let } I_a(C(X)) = (g_j \mid j \in J), \quad g_j \in I_a(C(X))$$

$$\Rightarrow g_j(x_0, x_1, \dots, x_n) = 0 \quad \forall (x_0, x_1, \dots, x_n) \in C(X)$$

$$\Rightarrow g_j(x_0 : x_1 : \dots : x_n) = 0 \quad \forall (x_0 : x_1 : \dots : x_n) \in X$$

$$\Rightarrow g_j \in I_p(X) \quad \forall j \in J \Rightarrow I_a(C(X)) \subseteq I_p(X) \quad \square$$

Corollary: $I_p(X)$ is always a radical ideal.

Theorem: If X is a projective variety, then $V_p(I_p(X)) = X$.

Proof: If $X = \emptyset$, then $I_p(X) = k[x_0, x_1, \dots, x_n]$ and $V_p(I_p(X)) = \emptyset$.

If $X \neq \emptyset$, then

$$\begin{aligned} V_p(I_p(X)) &= \mathbb{P} V_a(I_p(X)) \stackrel{\text{previous proposition}}{\downarrow} \mathbb{P} V_a(I_a(C(X))) \\ &= \mathbb{P}(C(X)) = X. \end{aligned} \quad \square$$

As in the affine case, we have:

Proposition: If $X \subseteq \mathbb{P}^n$ is any set, then $\bar{X} = V_p(I_p(X))$.

Affine weak Nullstellensatz: If $J \neq (1)$ is a proper ideal, then $V_a(J) \neq \emptyset$.

In projective case: $V_p(x_0, x_1, \dots, x_n) = \emptyset$

Definition: The ideal $I_0 = (x_0, x_1, \dots, x_n) \triangleleft k[x_0, x_1, \dots, x_n]$ is called the **irrelevant ideal**.

Theorem [projective weak Nullstellensatz]: For a proper homogeneous ideal $J \triangleleft k[x_0, x_1, \dots, x_n]$ we have $V_p(J) = \emptyset \Leftrightarrow \sqrt{J} = I_0$.

Proof: ($\Leftarrow$): Suppose $\sqrt{J} = I_0 = (x_0, x_1, \dots, x_n)$.

$\Rightarrow x_i \in \sqrt{J}$ for each $i \Rightarrow \forall i. \exists N_i$ s.t. $x_i^{N_i} \in J$.

Suppose that $(a_0 : a_1 : \dots : a_n) \in V_p(J)$.

$\Rightarrow \forall i. a_i^{N_i} = 0 \Rightarrow a_i = 0 \ \forall i \Rightarrow (0 : 0 : \dots : 0) \in V_p(J) \Rightarrow V_p(J) = \emptyset$
(not a projective point)

($\Rightarrow$): Suppose that $V_p(J) = \emptyset$. $J \neq (1) \Rightarrow V_a(J) \neq \emptyset$

J is a homogeneous ideal, so $J = (S)$ where S is a set of homogeneous polynomials. We know $V_p(J) = V_p(S)$, $V_a(J) = V_a(S)$, $V_p(S) = \mathbb{P} V_a(S) \Rightarrow V_p(J) = \mathbb{P} \underbrace{V_a(J)}_{\text{non-empty cone}}$

The only possibility is $V_a(J) = \{(0, 0, \dots, 0)\}$.

$\Rightarrow I_a(V_a(J)) = \sqrt{J} = (x_0, x_1, \dots, x_n)$. □

Theorem [projective Nullstellensatz]: For a homogeneous ideal $J \triangleleft k[x_0, x_1, \dots, x_n]$ with $\sqrt{J} \neq I_0$ we have $I_p(V_p(J)) = \sqrt{J}$.

Proof: If $V_p(J) = \emptyset$, then $J = (1)$ by the projective weak Nullstellensatz.

Then $\sqrt{J} = (1) = I_p(V_p(J)) = I_p(\emptyset)$.

Assume now that $V_p(J) \neq \emptyset$. J is a homogeneous ideal, so $J = (s)$ for some set s of homogeneous polynomials.

$$\begin{aligned} I_p(V_p(J)) &= I_p(V_p(s)) = I_a(\mathcal{C}(V_p(s))) = I_a(\mathcal{C}(\mathcal{P}(V_a(s)))) \\ &= I_a(V_a(s)) = I_a(V_a(J)) = \sqrt{J} \end{aligned}$$

↑ affine Nullstellensatz ▣

The maps I_p and V_p have similar properties as I_a and V_a . In some cases we need to assume that some ideal is not irrelevant.

Corollary: (1) We have a bijection:

$$\left\{ \begin{array}{l} \text{projective} \\ \text{varieties} \\ \text{in } \mathbb{P}^n \end{array} \right\} \xrightleftharpoons[V_p]{I_p} \left\{ \begin{array}{l} \text{homogeneous radical ideals} \\ \text{in } k[x_0, \dots, x_n] \text{ different from} \\ I_0 = (x_0, \dots, x_n) \end{array} \right\}$$

(2) We have a bijection:

$$\left\{ \begin{array}{l} \text{irreducible} \\ \text{projective} \\ \text{varieties} \\ \text{in } \mathbb{P}^n \end{array} \right\} \xrightleftharpoons[V_p]{I_p} \left\{ \begin{array}{l} \text{homogeneous prime ideals} \\ \text{in } k[x_0, \dots, x_n] \text{ different} \\ \text{from } I_0 = (x_0, \dots, x_n) \end{array} \right\}$$

Definition: Let $J \triangleleft k[x_0, \dots, x_n]$. The **homogenization** of J is the ideal generated by the homogenizations of all elements from J :

$$J^{(n)} = (f^{(n)} \mid f \in J) \triangleleft k[x_0, x_1, \dots, x_n].$$

Proposition: Let $X \subseteq \mathbb{A}^n$ be an affine variety, $\mathbb{A}^n \cong U_0 \subseteq \mathbb{P}^n$.

Let $J = I_a(X)^{(n)} \triangleleft k[x_0, \dots, x_n]$. Then $I_p(X) = J$ and $\bar{X} = V_p(J) = V_p(I_a(X)^{(n)})$.
Zariski closure

Proof: It is enough to prove that $J = I_p(X)$.

[$\subseteq$]: It is enough to show that $f^{(n)} \in I_p(X) \forall f \in I_a(X)$. Let $f \in I_a(X)$ be arbitrary. $I_a(X) \triangleleft k[x_1, \dots, x_n] \subseteq k[x_0, \dots, x_n]$. Let $a \in X$, $a = (a_0 : a_1 : \dots : a_n)$. Since $X \subseteq U_0$, we may assume that $a_0 = 1$.

$$\Rightarrow f(1, a_1, \dots, a_n) = 0 \quad (f \in I_a(X), a \in X)$$

$$f^{(n)}(a_0, a_1, \dots, a_n) = a_0^n f\left(\frac{a_1}{a_0}, \dots, \frac{a_n}{a_0}\right) = 0 \Rightarrow f^{(n)} \in I_p(X)$$

($\Rightarrow$): It is enough to show that $f \in J$ for each homogeneous f that vanishes on X .

Let $f \in I_p(X)$. If $f = x_0 \cdot g$, then also $g \in I_p(X)$, because $X \subseteq U_0$ and $x_0 \neq 0$ on X . So we assume that f is not divisible by x_0 . Let $a = (a_0, a_1, \dots, a_n) \in X$ be arbitrary.

We can assume $a_0 = 1$.

$$f \in I_p(X) \Rightarrow f(1, a_1, \dots, a_n) = 0$$

$\Rightarrow$ The polynomial $f(1, x_1, \dots, x_n)$ vanishes on X .

$$\Rightarrow f(1, x_1, \dots, x_n) \in I_a(X)$$

$$f = f(1, x_1, \dots, x_n)^{(n)} \Rightarrow f \in J$$

$\uparrow$ (not divisible by x_0)

Example: $X = V(x_1, x_2 - x_1^2)$

The ideal $(x_1, x_2 - x_1^2) = (x_1, x_2)$ is homogeneous.

$$\Rightarrow J^{(n)} = (x_1, x_2) \triangleleft k[x_0, x_1, x_2] \Rightarrow \bar{X} = V_p(x_1, x_2) = \{[1:0:0]\}.$$

Corollary: Let $X \subseteq \mathbb{P}^n$ be a projective variety and

$J_i = I_a(X \cap U_i)$ for $i = 0, \dots, n$; $J_i \triangleleft k[x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_n]$.

Then $I_p(X) = J_0^{(n)} \wedge J_1^{(n)} \wedge \dots \wedge J_n^{(n)}$.

Proof: $X = \bigcup_{i=0}^n (X \cap U_i) \quad / I_p$

$$I_p(X) = I_p\left(\bigcup_{i=0}^n (X \cap U_i)\right) = \bigcap_{i=0}^n I_p(X \cap U_i) \stackrel{\text{proposition}}{=} \bigcap_{i=0}^n I_a(X \cap U_i)^{(n)}$$

III. MAPS BETWEEN VARIETIES

1. Polynomial maps and coordinate ring

⚠ In this section everything will be affine.

Definition: Let $X \subseteq \mathbb{A}^n$ be an affine variety and $f \in k[x_1, \dots, x_n]$ a polynomial. The map $X \rightarrow k, a \mapsto f(a)$ is called a **polynomial function** on X .

The set of all polynomial functions on X is a ring for point-wise addition and multiplication. We call it the **coordinate ring** of X . Notation: $k[X]$

Proposition: Let $X \subseteq \mathbb{A}^n$ be an affine variety. Then $k[X] \cong k[x_1, \dots, x_n] / I(X)$.

Proof: Let $\Phi: k[x_1, \dots, x_n] \rightarrow k[X]$ be the map defined by $f \mapsto (a \mapsto f(a))$. Φ is a ring homomorphism and it is clearly surjective, so $k[X] \cong k[x_1, \dots, x_n] / \ker \Phi$.
 $f \in \ker \Phi \Leftrightarrow f(a) = 0 \forall a \in X \Leftrightarrow f(a) \in I(X)$.
 $\Rightarrow k[X] \cong k[x_1, \dots, x_n] / I(X)$. □

Corollary: $k[\mathbb{A}^n] = k[x_1, \dots, x_n]$ $I(\mathbb{A}^n) = (0)$

Proposition:

- (1) $k[X]$ is without nilpotents. We say it is **reducible**.
- (2) $k[X]$ is a domain $\Leftrightarrow X$ is irreducible.

Proof: (1) Some power of a function is 0 $\Leftrightarrow$ the function is 0.
(2) $k[X] \cong k[x_1, \dots, x_n] / I(X)$ is a domain $\Leftrightarrow I(X)$ is a prime ideal
 $\Leftrightarrow X$ is irreducible. □

Remark: If $X = X_1 \cup \dots \cup X_m$ is the decomposition of X into irreducible components, then $k[X] \cong k[x_1] \times \dots \times k[x_m]$.

Commutative algebra: Chinese Remainder Theorem

Definition: Let $X \subseteq A^n$ be a variety.

(1) A **subvariety** of X is any subset of the form

$$V_X(S) := \{a \in X \mid f(a) = 0 \ \forall f \in S\}$$

where $S \subseteq k[X]$.

(2) For any subset $Y \subseteq X$ we define the **ideal of Y** in $k[X]$ by $I_X(Y) = \{f \in k[X] \mid f(a) = 0 \ \forall a \in Y\} \triangleleft k[X]$.

The maps I_X and V_X have the following properties:

(1) If $S \subseteq k[X]$ and $J \triangleleft k[X]$ is the ideal generated by S , then $V_X(S) = V_X(J)$.

(2) $k[X]$ is a quotient of a noetherian ring, so it is noetherian $\Rightarrow$ subvarieties of X are of the form $V_X(S)$ for finite S .

(3) Subvarieties of X are precisely the varieties that are contained in X .

(4) If Y is a subvariety of X , then

$$k[X]_{I_X(Y)} \cong (k[x_1, \dots, x_n] / I(X)) / (I(Y) / I(X)) \stackrel{\text{isomorphism theorem}}{\cong} k[x_1, \dots, x_n]_{I(Y)} \cong k[Y]$$

(5) $V_X(I_X(Y)) = Y$ if Y is a subvariety of X .

(6) **Relative Nullstellensatz**:

If $J \triangleleft k[X]$, then $I_X(V_X(J)) = \sqrt{J}$.

(7) Versions of properties from the proposition with 16 properties for V and I hold.

(8) There is a bijective correspondence between subvarieties of X and radical ideals of $k[X]$.

Definition: Let $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^m$ be affine varieties. A map $\phi: X \rightarrow Y$ is a **polynomial map** if there exist polynomials $f_1, \dots, f_m \in k[x_1, \dots, x_n]$ such that $\phi(a) = (f_1(a), \dots, f_m(a))$ for each $a \in X$.

Lemma: Polynomial maps are continuous in the Zariski topology.

Proof: Let $X \subseteq \mathbb{A}^n$, $Y \subseteq \mathbb{A}^m$ and $\phi: X \rightarrow Y$ a polynomial map. Then there exist polynomials $f_1, \dots, f_m \in k[x_1, \dots, x_n]$ s.t. $\phi(a) = (f_1(a), \dots, f_m(a)) \forall a \in X$. Let $Z \subseteq Y$ be a closed subset. We have to prove that $\phi^{-1}(Z)$ is closed. Z is closed in Y , which is closed in $\mathbb{A}^m$ so Z is closed in $\mathbb{A}^m$, so Z is an affine variety. Therefore there exist $g_1, \dots, g_\ell \in k[x_1, \dots, x_m]$ s.t. $Z = V(g_1, \dots, g_\ell)$.

$$\begin{aligned} \phi^{-1}(Z) &= \{a \in X \mid \phi(a) \in Z\} \\ &= \{a \in X \mid g_i(\phi(a)) = 0 \forall i = 1, \dots, \ell\} \\ &= \{a \in X \mid g_i(f_1(a), \dots, f_m(a)) = 0 \forall i\} \end{aligned}$$

Observe that $g_i(f_1(x_1, \dots, x_n), \dots, f_m(x_1, \dots, x_n))$ is a polynomial for each i . So $\phi^{-1}(Z)$ is an affine variety, so closed in X . ▢

Corollary: If $\phi: X \rightarrow Y$ is a polynomial map and $S \subseteq X$ any subset, then $\phi(\overline{S}) = \overline{\phi(S)}$.

Corollary: If X is an irreducible variety and $\phi: X \rightarrow Y$ is a polynomial map, then $\overline{\phi(X)}$ is irreducible.

Proof: Assume that $\overline{\phi(X)} = Z_1 \cup Z_2$ for two closed subsets $Z_1, Z_2 \subseteq \overline{\phi(X)}$. If $a \in X$ is arbitrary, then $\phi(a) \in \overline{\phi(X)} = Z_1 \cup Z_2$, so $\phi(a) \in Z_i$ for some $i \in \{1, 2\}$. We showed that $X \subseteq \phi^{-1}(Z_1 \cup Z_2)$.
 $a \in \phi^{-1}(Z_i)$

Since ϕ is continuous, $\phi^{-1}(Z_1)$ and $\phi^{-1}(Z_2)$ are closed, and irreducibility of X implies $X \subseteq \phi^{-1}(Z_1)$ for some $i \in \{1, 2\}$.

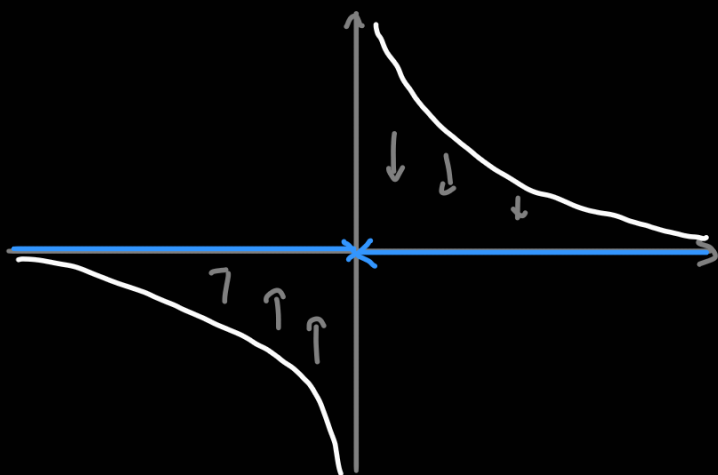
$$\Rightarrow \phi(X) \subseteq \phi(\phi^{-1}(Z_i)) \subseteq Z_i$$

Z_i is closed, so $\overline{\phi(X)} \subseteq Z_i \Rightarrow Z_i$ is not a proper subset of $\overline{\phi(X)} \Rightarrow \phi(X)$ is irreducible $\square$

Corollary: If $\phi: A^n \rightarrow X$ is a polynomial map, then $\phi(A^n)$ is irreducible.

The image of a polynomial map is not necessarily closed.

Example: $X = V(xy - 1)$, $\phi: X \rightarrow A^1$ projection



$$\phi(X) = A^1 \setminus \{0\}$$

The image of a polynomial map is also not necessarily open.

Example: $\phi: A^2 \rightarrow A^2$

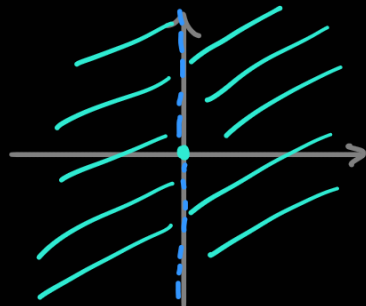
$$(x, y) \mapsto (x, xy)$$

What is $\phi(A^2)$?

If $x \neq 0$, we can get all pairs of the form (x, z) , $z \in A^1$.

If $x = 0$, we get only $(0, 0)$.

$$\Rightarrow \phi(A^2) = (A^1 \setminus \{0\}) \times A^1 \cup \{(0, 0)\}.$$



This is not open in A^2 .

Definition: Affine varieties X and Y are **isomorphic** if there exist polynomial maps $\phi: X \rightarrow Y$ and $\tau: Y \rightarrow X$ s.t. $\phi \circ \tau = \text{id}_Y$ and $\tau \circ \phi = \text{id}_X$.

Bijective polynomial maps are not necessarily isomorphisms.

Example: $X = V(x^2 - y^3)$, $\phi: A^1 \rightarrow X$
 $t \mapsto (t^3, t^2)$

$\phi(A^1)$ indeed lies in X : $(t^3)^2 - (t^2)^3 = 0$.

ϕ is a polynomial map and it is injective:

If $(t^3, t^2) = (u^3, u^2)$ and $t \neq 0$, then $\left. \begin{matrix} t^3 = u^3 \\ t^2 = u^2 \end{matrix} \right\} \Rightarrow t = u$

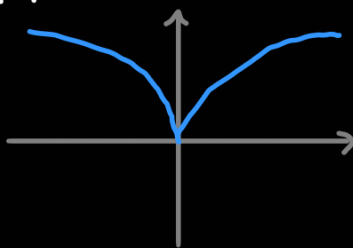
The map ϕ is also surjective: Let $(a, b) \in X$ and suppose $a, b \neq 0$.

$$\phi\left(\frac{a}{b}\right) = \left(\frac{a^3}{b^3}, \frac{a^2}{b^2}\right) = \left(\frac{a^3}{a^2}, \frac{b^3}{b^2}\right) = (a, b)$$

$\hat{a} \quad (a, b) \in X \Rightarrow a^2 = b^3$

$\Rightarrow \phi$ is a bijective polynomial map, but it is not an isomorphism (proof later).

X has a cusp in $(0, 0)$,
 but A^1 is smooth



Lemma: Composition of polynomial maps is a polynomial map.

Proof: $\phi: X \rightarrow Y$, $\tau: Y \rightarrow Z$, $X \subseteq A^n$, $Y \subseteq A^m$, $Z \subseteq A^l$,
 ϕ, τ polynomial maps.

$\Rightarrow \exists f_1, \dots, f_m \in k[x_1, \dots, x_n]$ s.t. $\phi(a) = (f_1(a), \dots, f_m(a)) \quad \forall a \in X$.

$\exists g_1, \dots, g_l \in k[x_1, \dots, x_m]$ s.t. $\tau(u) = (g_1(u), \dots, g_l(u)) \quad \forall u \in Y$.

$$\begin{aligned} \forall a \in X : (\tau \circ \phi)(a) &= \tau(f_1(a), \dots, f_m(a)) \\ &= (g_1(f_1(a), \dots, f_m(a)), \dots, g_l(f_1(a), \dots, f_m(a))) \end{aligned}$$

Components are polynomials, as they are compositions of polynomials. $\square$

Lemma: Let $X \subseteq \mathbb{A}^n$, $Y \subseteq \mathbb{A}^m$, $\phi: X \rightarrow Y$ a map and $\pi_i: Y \rightarrow \mathbb{A}^1$ be the projection to the i -th component. ϕ is a polynomial map $\Leftrightarrow$ all compositions $\pi_i \circ \phi$ are polynomial functions.

Proof: $(\Rightarrow)$: Follows from the previous lemma.

$(\Leftarrow)$: Suppose that $\pi_i \circ \phi$ is a polynomial function for each i . Then $\forall i$ $\exists f_i \in k[x_1, \dots, x_n]$ s.t. $\pi_i(\phi(a)) = f_i(a) \forall a \in X$.

$\Rightarrow \phi(a) = (f_1(a), \dots, f_m(a)) \Rightarrow \phi$ is a polynomial map. $\square$

Corollary: Let $\phi: X \rightarrow Y$ be a polynomial map and $g \in k[Y]$. Then $g \circ \phi \in k[X]$. $X \xrightarrow{\phi} Y \xrightarrow{g} k$

Definition: Let ϕ and g be as in the corollary. The element $g \circ \phi \in k[X]$ is called the **pullback** (slovene: povlek) of g under ϕ . We will denote it by $\phi^*(g)$.

Let $\phi: X \rightarrow Y$ be a polynomial map. Then we have a map $\phi^*: k[Y] \rightarrow k[X]$.

Lemma: ϕ^* is a homomorphism of k -algebras

Lemma: If $\phi: X \rightarrow Y$ and $\tau: Y \rightarrow Z$ are polynomial maps, then $(\tau \circ \phi)^* = \phi^* \circ \tau^*$.

Theorem: The map $\phi \rightarrow \phi^*$ gives a bijection between the set of polynomial maps $X \rightarrow Y$ and the set of k -algebra homomorphisms $k[Y] \rightarrow k[X]$.

Proof: $\phi^*: k[Y] \rightarrow k[X]$ is an algebra homomorphism by the lemma. We have to prove bijectivity.

Injectivity: Suppose $\phi, \psi: X \rightarrow Y$ are polynomial maps s.t.
 $\phi^* = \psi^*: k[Y] \rightarrow k[X]$.

$$\begin{aligned}\phi^*(g) &= \psi^*(g) & \forall g \in k[Y] \\ g \circ \phi &= g \circ \psi & \forall g \in k[Y] \\ g(\phi(a)) &= g(\psi(a)) & \forall g \in k[Y], \forall a \in X\end{aligned}$$

Let $f_1, \dots, f_m \in k[x_1, \dots, x_n]$ s.t. $\phi(a) = (f_1(a), \dots, f_m(a)) \quad \forall a \in X$
 $h_1, \dots, h_m \in k[x_1, \dots, x_n]$ s.t. $\psi(a) = (h_1(a), \dots, h_m(a)) \quad \forall a \in X$

$$\Rightarrow g(f_1(a), \dots, f_m(a)) = g(h_1(a), \dots, h_m(a)) \quad \forall g \in k[Y] \quad \forall a \in X$$

For g we take projection to the i -th component:

$$f_i(a) = h_i(a) \quad \forall a \in X, \forall i$$

$$\Rightarrow (f_1(a), \dots, f_m(a)) = (h_1(a), \dots, h_m(a)) \quad \forall a \in X \Rightarrow \phi = \psi$$

Surjectivity: Let $F: k[Y] \rightarrow k[X]$ be a homomorphism of k -algebras. We have to show that there is a polynomial map $\phi: X \rightarrow Y$ s.t. $F = \phi^*$.

$$X \subseteq A^n, Y \subseteq A^m$$

$$\text{Then } k[Y] = k[x_1, \dots, x_m] / I(Y).$$

For $i=1, \dots, m$ denote $\bar{x}_i = x_i + I(Y) \in k[Y]$. As a function on Y , $\bar{x}_i$ is the projection to the i -th component.

Define $g_i := F(\bar{x}_i) \in k[X]$ for $i=1, \dots, m$. Consider g_i as polynomial functions on X

$$\begin{aligned}g: X &\rightarrow k \\ a &\mapsto g_i(a)\end{aligned}$$

$$\text{Define } \phi: X \rightarrow A^m$$

$$a \mapsto (g_1(a), \dots, g_m(a))$$

$$\phi(X) \subseteq Y$$

Let $h \in I(Y)$ and $a \in X$ arbitrary. We have to show that $h(\phi(a)) = 0$.

$$h(\phi(a)) = h(g_1(a), \dots, g_m(a)) = h(g_1, \dots, g_m)(a) = 0$$

$$h(g_1, \dots, g_m) = h(F(\bar{x}_1), \dots, F(\bar{x}_m))$$

F homomorphism of algebras, h is a polynomial

$$\rightarrow = F(h(\bar{x}_1, \dots, \bar{x}_m))$$

$$= F(\underbrace{h(x_1, \dots, x_m) + I(Y)}_{\in I(Y)}) = 0$$

$$\underbrace{\in I(Y)}_{=0 \text{ in } k[x_1, \dots, x_m]/I(Y) = k[Y]}$$

$$= 0 \text{ in } k[x_1, \dots, x_m]/I(Y) = k[Y]$$

$$F = \phi^* \Leftrightarrow F(f) = \phi^*(f) \quad \forall f \in k[Y].$$

F and ϕ^* are algebra homomorphisms, therefore it is enough to check the equality $F(f) = \phi^*(f)$ on the generators of $k[Y]$, i.e. on $\bar{x}_1, \bar{x}_2, \dots, \bar{x}_m$.

$$F(\bar{x}_i) = \phi^*(\bar{x}_i)$$

For each $a \in X$ we have $\phi^*(\bar{x}_i)(a) = \bar{x}_i(\phi(a))$

$$\Rightarrow \bar{x}_i(g_1(a), \dots, g_m(a)) = g_i(a) = F(\bar{x}_i)(a).$$

$\uparrow$ projection to i -th component

$$\Rightarrow \phi^*(\bar{x}_i) = F(\bar{x}_i).$$



Corollary: There is a contravariant functor

$$\{\text{affine varieties}\} \longrightarrow \{\text{finitely generated reduced } k\text{-algebras}\}$$

morphisms: polynomial maps

$$\text{On objects: } X \longmapsto k[X]$$

$$\text{On morphisms: } (\phi: X \rightarrow Y) \longrightarrow (\phi^*: k[Y] \longrightarrow k[X]).$$

Moreover, the following version of surjectivity holds: If A is any finitely generated reduced k -algebra, then there exists an affine variety X s.t. $k[X] \cong A$.

Proof: We already proved all properties of the functor.

Let A be a finitely generated reduced k -algebra. Let A be

generated by $a_1, \dots, a_n$. Then we have an algebra homomorphism $k[x_1, \dots, x_n] \rightarrow A$, $x_i \mapsto a_i \forall i$.

$\Rightarrow A \cong k[x_1, \dots, x_n]/I$ for some ideal I

A is reduced $\Rightarrow I$ is a radical ideal

By the Nullstellensatz: $I = I_A(X)$ for some affine variety

$X \subseteq \mathbb{A}^n$

$\stackrel{v(I)}{=} I$

$\Rightarrow A \cong k[x_1, \dots, x_n]/I(X) = k[X]$



Remark: The above Functor induces a Functor:

$\left\{ \begin{array}{l} \text{isomorphism classes} \\ \text{of affine varieties} \end{array} \right\} \longrightarrow \left\{ \begin{array}{l} \text{isomorphism classes of} \\ \text{finitely generated reduced} \\ \text{k-algebras} \end{array} \right\}$

which is bijective on objects.

Another corollary of the theorem:

Corollary: Let $\phi: X \rightarrow Y$ be a polynomial map. Then ϕ is an isomorphism $\Leftrightarrow \phi^*: k[Y] \rightarrow k[X]$ is an isomorphism.

Proof: ($\Rightarrow$): Suppose ϕ is an isomorphism. Then $\exists \tau: Y \rightarrow X$ polynomial map s.t. $\phi \circ \tau = \text{id}_Y$ and $\tau \circ \phi = \text{id}_X$.

By one of the lemmas before the theorem:

$$\phi^* \circ \tau^* = (\tau \circ \phi)^* = \text{id}_X^* = \text{id}_{k[X]}$$

The same argument shows $\tau^* \circ \phi^* = \text{id}_{k[Y]} \Rightarrow \phi^*$ is an isomorphism of algebras.

($\Leftarrow$): Assume that $\phi^*: k[Y] \rightarrow k[X]$ is an isomorphism. Then $\exists$ an algebra homomorphism $F: k[X] \rightarrow k[Y]$ s.t. $\phi^* \circ F = \text{id}_{k[X]}$, $F \circ \phi^* = \text{id}_{k[Y]}$.

By surjectivity in the theorem $\exists$ a polynomial map

$\tau: Y \longrightarrow X$ s.t. $F = \tau^*$.

$$(\phi \circ \tau)^* = \tau^* \circ \phi^* = F \circ \phi^* = \text{id}_{k[Y]}$$

Also: $\text{id}_Y^* = \text{id}_{k[Y]}$. By the injectivity of the theorem $\phi \circ \tau = \text{id}_Y$. The same argument gives $\tau \circ \phi = \text{id}_X$
 $\Rightarrow \phi$ is an isomorphism of varieties. 

7. november 2025