# V. DIMENZIJE RAZNOTEROSTI

## 1. Definicija in osnovne lastnosti

**Definicija:** (1) Naj bo $X$ kvaziprojektivna raznoterost. Dimenzija raznoterosti $X$ je

$$\dim X := \sup\{n \mid \text{obstaja veriga nerazcepnih zaprtih podmnožic v } X : \emptyset \neq X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_n \subseteq X\}$$

(2) Naj bo $Y \neq 0$ nerazcepna zaprta podmnožica v $X$. Kodimenzija od $Y$ v $X$ je

$$\operatorname{codim}_X Y := \sup\{n \mid \text{obstaja veriga nerazcepnih zaprtih podmnožic v } X : Y \subseteq Y_0 \subsetneq Y_1 \subsetneq \cdots \subsetneq Y_n \subseteq X\}$$

Dimenzija raznoterosti je element $\mathbb{N}_0 \cup \{\infty\}$.

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**Lemma:** Let $X$ be a quasiprojective variety and $Y$ a closed subset of $X$. Then $\dim Y \leq \dim X$.

**Proof:** If $\emptyset = Z_0 \subsetneq Z_1 \subsetneq Z_2 \subsetneq \cdots \subsetneq Z_n \subseteq Y$ is a chain of irreducible closed subsets of $Y$, then this is also a chain of irreducible closed subsets of $X$. $\blacksquare$

**Lemma:** If $X$ is a quasiprojective variety and $X = X_1 \cup \cdots \cup X_m$ is the decomposition into irreducible components, then $\dim X = \max\{\dim X_1, \ldots, \dim X_r\}$.

**Proof:** Suppose $\emptyset \neq Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n \subseteq X$ is a chain of irreducible closed subsets of $X$. Then we can write

$$Z_n = (Z_n \cap X_1) \cup (Z_n \cap X_2) \cup \cdots \cup (Z_n \cap X_m).$$

$Z_n$ is irreducible, so $Z_n \subseteq X_i$ for some $i$. But then $\emptyset \neq Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n$ is a chain of irreducible closed subsets of $X_i$. $\Rightarrow \dim X \leq \dim X_i$ for some $i$.

The inequality $\dim X_i \leq \dim X \;\forall i$ holds by the previous lemma, so we have equality for some $i$. $\blacksquare$

<u>Example</u>: An irreducible variety of dimension $0$ is a point, and each point is an irreducible variety of dimension $0$:
Suppose $X$ is irreducible and not a point, then $\{a\} \subsetneq X$ is a chain of irreducible subvarieties for some $a \in X \Rightarrow \dim X \geq 1$.
If $X = \{a\}$, then it is clear that $\dim X = 0$.

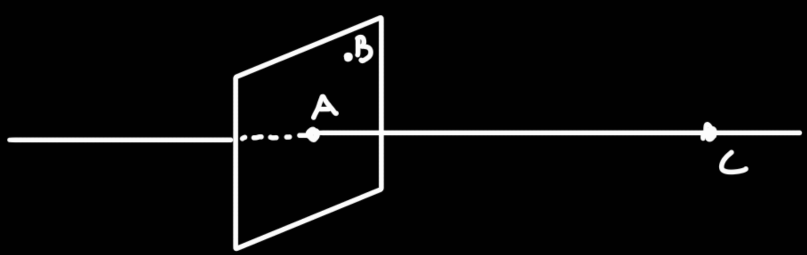
By the lemma it follows that a variety has dimension $0 \Leftrightarrow$ it is a finite union of points.

<u>Definition:</u> A quasiprojective variety is of pure dimension $d$ if each of its irreducible components has dimension $d$.
slovene: raznoterost čiste dimenzije

<u>Example</u>: $X = \underset{\substack{\text{plane} \\ \text{of dim } 2}}{\underset{\uparrow}{V(x)}} \cup \underset{\substack{\text{line of} \\ \text{dim } 1}}{\underset{\uparrow}{V(y,z)}} \subseteq \mathbb{A}^3 \Rightarrow X$ is not of pure dimension

<u>Definition:</u> Let $X$ be a quasiprojective variety and $a \in X$. The local dimension of $X$ at $a$ is the maximum of dimensions of those irreducible components of $X$ that contain $a$. We denote it by $\dim_a X$.

<u>Example</u>:




$\dim_A X = 2$
$\dim_B X = 2$
$\dim_C X = 1$

**Definition:** (1) The Krull dimension of a ring $R$ (commutative and with identity) is
$\dim R = \sup\{n \mid P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_n \subsetneq R$ is a chain of prime ideals of $R\}$.
(2) Let $I \trianglelefteq R$ be a prime ideal. The codimension or height of $I$ is $\operatorname{codim}_R I = \operatorname{ht} I := \sup\{n \mid P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_n \subseteq I$ is a chain of prime ideals in $R\}$.
(3) If $I \trianglelefteq R$ is an arbitrary ideal, then the codimension or height of $I$ is defined as $\operatorname{codim}_R I = \operatorname{ht} I = \min\{\operatorname{ht} P \mid I \subseteq P,\ P \text{ prime ideal}\}$.

**Examples:** 1) The dimension of a field is 0.
2) Principal ideal domains are of dimension 1.

**Lemma:** Let $X$ be an affine variety and $Y$ an irreducible subvariety of $X$. Then
(1) $\dim X = \dim \mathbb{k}[X]$
(2) $\operatorname{codim}_X X = \operatorname{codim}_{\mathbb{k}[X]} I_X(Y)$.

**Proof:** (1) $\dim X = \sup\{n \mid \emptyset \neq Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n \subseteq X$ is a chain of irreducible subvarieties$\}$

irreducible varieties correspond to prime ideals

$= \sup\{n \mid \mathbb{k}[x_1,\dots,x_n] = P_0 \supsetneq P_1 \supsetneq \cdots \supsetneq P_n = I(X)$ is a chain of prime ideals$\}$

prime ideals in the quotient

$= \sup\{n \mid \mathbb{k}[x_1,\dots,x_m]/I(X) \neq Q_0 \supsetneq Q_1 \supsetneq \cdots \supsetneq Q_n$ is a chain of prime ideals in $\mathbb{k}[X]\}$
$= \dim \mathbb{k}[X]$.

(2) $Y$ is irreducible $\Rightarrow I_X(Y)$ is a prime ideal in $\mathbb{k}[X]$.
$\operatorname{codim}_X Y = \sup\{n \mid Y \subseteq Y_0 \subsetneq Y_1 \subsetneq \cdots \subsetneq Y_n \subseteq X \subseteq \mathbb{A}^m;\ Y_i$ irreducible varieties in $\mathbb{A}^m\}$
$= \sup\{n \mid I(Y) \supseteq P_0 \supsetneq P_1 \supsetneq \cdots \supsetneq Y_n \supseteq I(X);\ P_i$ prime ideals in $\mathbb{k}[x_1,\dots,x_m]\}$

quotient

$= \sup\{n \mid \underbrace{I(X)/I(Y)}_{I_X(Y)} \supseteq Q_0 \supsetneq Q_1 \supsetneq \cdots \supseteq Q_n;\ Q_i$ prime ideals in $\mathbb{k}[X]\}$
$= \operatorname{codim}_{\mathbb{k}[X]} I_X(Y)$ ∎

A similar proof shows:
**Proposition:** $\dim_a X = \dim \mathcal{O}_{X,a}$.

## 2.1. Algebraic results on dimension and their geometric consequences

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**Theorem:** If $R$ is a noetherian ring, then $\dim R[x] = \dim R + 1$.

**Corollary:** $\dim \Bbbk[x_1, \dots, x_n] = n$.

**Proposition:** $\dim \mathbb{A}^n = n$

**Proof:** $\dim \mathbb{A}^n = \dim \Bbbk[\mathbb{A}^n] = \dim \Bbbk[x_1, \dots, x_n] = n$. ◪

**Corollary:** The dimension of each affine variety is finite. If $X$ is an affine variety in $\mathbb{A}^n$, then $\dim X \leq n$.

**Definition:** Let $F \subseteq E$ be a field extension.
(1) Elements $a_1, \dots, a_n \in E$ are *algebraically independent* over $F$ if there exists no nonzero polynomial $f \in F[x_1, \dots, x_n]$ such that $f(a_1, \dots, a_n) = 0$.
(2) The *transcendence degree* of $E$ over $F$ is the maximal possible number of elements of $E$ that are algebraically independent over $F$. We denote it by $\mathrm{Trdeg}_F E$. transcendenčna stopnja

**Theorem:** Let $R$ be a finitely generated domain over $\Bbbk$. Then $\dim R = \mathrm{Trdeg}_{\Bbbk} F$ where $F$ is the field of fractions of $R$.

**Proposition:** Let $X$ be an irreducible affine variety. Then $\dim X = \mathrm{Trdeg}_{\Bbbk} \Bbbk(X)$.

**Proof:** $\dim X = \dim \Bbbk[X]$
$X$ is irreducible $\Rightarrow \Bbbk[X]$ is a domain. It is also finitely generated. Apply the previous theorem. ◪

Corollary [important]: Birationally equivalent irreducible varieties have the same dimension.

Theorem: Let $R$ be a finitely generated domain over $\Bbbk$. Then every maximal (with respect to inclusion) chain of prime ideals in $R$ has length $\dim R$ (it has $\dim R+1$ prime ideals and $\dim R$ inclusions).

Remark: The theorem does not hold for general noetherian rings. It is important that $R$ is a finitely generated domain over a field.

Theorem: Let $X$ be an irreducible affine or projective variety and $Y \neq \emptyset$ its irreducible subvariety. Then $\dim X = \dim Y + \operatorname{codim}_X Y$. In pardicular, if $a \in X$, then $\dim X = \operatorname{codim}_X \{a\}$.

Proof: We prove the statement for affine $X$. The proof in the projective case is similar, we have to use $S[X]$ instead of $\Bbbk[X]$. Let $n = \dim Y$ and $m = \operatorname{codim}_X Y$. Then there exist chains of irreducible affine varieties

$$\emptyset \neq X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_n \subseteq Y$$
$$Y \subseteq Y_0 \subsetneq Y_1 \subsetneq \cdots \subsetneq Y_m \subseteq X$$

$X$ and $Y$ are irreducible, so maximality implies $X_n = Y = Y_0$ and $Y_m = X$. So, we have a chain of irreducible affine varieties

$$\emptyset \neq X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_n = Y \subsetneq Y_1 \subsetneq \cdots \subsetneq Y_m = X,$$

which is maximal with respect to inclusion. We apply the map $I_X$ to get a chain of prime ideals

$$\Bbbk[X] = I_X(X_0) \supsetneq I_X(X_1) \supsetneq \cdots \supsetneq I_X(X_n) \supsetneq I_X(Y_1) \supsetneq \cdots \supsetneq I_X(Y_m) = I_X(X) = (0),$$

which is maximal with respect to inclusion.

By the algebraic theorem we get $\dim \Bbbk[X] = n+m \Rightarrow \dim X = n+m$. ▨

It is important that $X$ is irreducible.

**Example:** $X = V(x) \cup V(y,z) \subseteq \mathbb{A}^3$ $\Rightarrow \dim X = 2$. (reducible)

$Y = V(y,z) \Rightarrow \dim Y = 1$

$Y$ is an irreducible component of $X$, so there is no irreducible variety between $Y$ and $X$ $\Rightarrow \operatorname{codim}_X Y = 0$.

$\Rightarrow \dim X \neq \dim Y + \operatorname{codim}_X Y$.

**Corollary:** If $X$ is an irreducible affine or projective variety and $Y$ is its proper irreducible subvariety, then $\dim Y < \dim X$.

**Proof:** $Y \subsetneq X$ is a chain of irreducible varieties $\Rightarrow \operatorname{codim}_X Y \geq 1$. Now apply the theorem. $\blacksquare$

If $X$ is reducible, then we can have $\dim Y = \dim X$ if $Y$ is an irreducible component of $X$ of largest dimension.

**Corollary:** If $X \subseteq \mathbb{P}^n$ is a projective variety, then $\dim C(X) = 1 + \dim X$.

**Proof:** We can assume that $X$ is irreducible. Let $n = \dim X$. Then $\exists$ a chain of irreducible varieties: $\emptyset \neq X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_m \subseteq X$.

Then $\emptyset \neq \{0\} \subsetneq C(X_0) \subsetneq \cdots \subsetneq C(X_m) \subseteq C(X)$ is a chain of irreducible affine varieties. $\Rightarrow \dim C(X) \geq 1 + \dim X$

We do the same for the codimension to get $\operatorname{codim}_{\mathbb{A}^{n+1}} C(X) \geq \operatorname{codim}_{\mathbb{P}^n} X$.

By the theorem we get $n+1-\dim C(X) \geq n - \dim X \Leftrightarrow \dim C(X) \leq 1 + \dim X$

$\Rightarrow \dim C(X) = 1 + \dim X$. $\blacksquare$

**Remark:** What we proved for affine varieties also holds for projective varieties.

**Corollary:** $\dim \mathbb{P}^n = n$

**Lemma:** Let $X$ be an affine or projective variety, irreducible, and let $V$ be a nonempty open subset of $X$. Then $\dim V = \dim X$.

**Proof:** Assume that $n = \dim U$. Then there exists a chain of irreducible closed subsets of $U$: $\emptyset \neq Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n \subseteq U$
For each $i$ let $\overline{Z_i}$ be the closure of $Z_i$ in $X$. Then
$\emptyset \neq \overline{Z}_0 \subsetneq \overline{Z}_1 \subsetneq \cdots \subsetneq \overline{Z}_n \subseteq X$ is a chain of irreducible subvarieties of $X \Rightarrow \dim X \geq n$.
Conversely, assume $\dim X = n$ and let $a \in U$. Then $\operatorname{codim}\{a\} = n$ and there exists a chain of irreducible subvarieties
$\emptyset \neq \{a\} = X_0 \subsetneq X_1 \subseteq \cdots \subseteq X_n \subseteq X$. Then $\emptyset \subseteq \{a\} \subsetneq X_1 \cap U \subsetneq \cdots \subseteq X_n \cap U \subseteq U$ is a chain of irreducible closed subsets of $U$.
If $X_i \cap U = X_{i+1} \cap U \Rightarrow \overline{X_i \cap U} = \overline{X_{i+1} \cap U}$
(using irreducibility of $X_j$) $\quad \overset{\shortparallel}{X_i} \neq \overset{\shortparallel}{X_{i+1}}$
$\Rightarrow$ inclusions are strict $\Rightarrow \dim U \geq n$ $\blacksquare$

**Corollary:** Let $X$ be an affine or projective variety and $U$ a dense open subset. Then $\dim V = \dim X$.

**Corollary:** If $X$ is an affine variety and $\overline{X}$ the projective closure, then $\dim X = \dim \overline{X}$.

**Corollary:** $\dim(\mathbb{P}^n \times \mathbb{P}^m) = n+m$

**Proof:** $\mathbb{P}^n \times \mathbb{P}^m$ contains the open sets $\mathbb{A}^n \times \mathbb{A}^m$. $\blacksquare$

**Remark:** The last corollary holds more generally. If $X$ and $Y$ are any varieties, then $\dim X \times Y = \dim X + \dim Y$.

The proof of the remark is left as an exercise.

**Corollary:** The image of the Segre embedding

$$\sigma_{m,n}: \mathbb{P}^m \times \mathbb{P}^n \longrightarrow \mathbb{P}^{(m+1)(n+1)-1}$$

$$((x_0:x_1:\ldots:x_m),(y_0:y_1:\ldots:y_n)) \longmapsto (x_0y_0 : x_0y_1 : x_0y_2 : \cdots : x_my_n)$$

has dimension $m+n$.

**Proof:** $\sigma_{m,n}$ is an isomorphism to its image. $\square$

**Corollary:** The Veronese variety $V_d(\mathbb{P}^n)$ has dimension $n$.

$$V_d : \mathbb{P}^n \longrightarrow \mathbb{P}^N \text{ Veronese map}$$

$d=2, n=2$: $(x_0:x_1:x_2) \longmapsto (x_0^2 : x_0x_1 : x_0x_2 : x_1^2 : x_1x_2 : x_2^2)$

$d=3, n=2$: $(x_0:x_1:x_2) \longmapsto (x_0^3 : x_0^2x_1 : x_0^2 x_2 : x_0x_1^2 : \ldots)$

**Corollary:** $\dim Gr(\ell,n) = \ell(n-\ell)$

**Proof:** We covered $Gr(\ell,n)$ with open subsets isomorphic to $\mathbb{A}^{\ell\times(n-\ell)}$. $\square$

**Recall:** If $F: R \longrightarrow S$ is a homomorphism of rings, then $S$ becomes an $R$-module for the action $r s := F(r)s$.
In particular, if $R$ is a subring of $S$, then $S$ is an $R$-module.

**Definition:** Let $R$ be a subring of $S$. An element $s\in S$ is integral (celosten) over $R$ if there exist $n\in\mathbb{N}$ and $a_0, a_1, \ldots, a_{n-1} \in R$ such that $s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0 = 0$.
$\Leftrightarrow$ $s$ is a root of a monic polynomial with coefficients in $R$.

**Proposition:** Let $S$ be a finitely generated $R$-algebra. Then $S$ is integral over $R$ (i.e. every element of $S$ is integral over $R$) $\Leftrightarrow$ $S$ is finitely generated as an $R$-module.

Let $\phi: X \longrightarrow Y$ be a dominant regular map between affine varieties. Then $\phi^*: \mathbb{k}[Y] \longrightarrow \mathbb{k}[X]$ is an injective homomorphism of rings.

$\Rightarrow$ We may view $\mathbb{k}[Y]$ as a subring of $\mathbb{k}[X]$. Also, $\mathbb{k}[X]$ is a module over $\mathbb{k}[Y]$.

<u>Definition</u>: (1) Let $\phi: X \longrightarrow Y$ be a dominant regular map between affine varieties $X$ and $Y$. $\phi$ is a finite map or finite morphism if $\mathbb{k}[X]$ is a finitely generated $\mathbb{k}[Y]$-module ($\Leftrightarrow$ every element of $\mathbb{k}[X]$ is integral over $\mathbb{k}[Y]$).

(2) Let $\phi: X \to Y$ be a dominant regular map between quasiprojective varieties. $\phi$ is a finite morphism if for each $a \in Y$ there exists an affine open neighbourhood $V$ of $a$ such that $U := \phi^{-1}(V)$ is also affine and $\phi|_U : U \to V$ is a finite map according to (1).

<u>Proposition</u>: Finite morphism has finite fibers.

<u>Proof</u>: It is enough to check this in the affine case. Let $\phi: X \longrightarrow Y$ be a finite morphism, $X \subseteq \mathbb{A}^n$, $Y \subseteq \mathbb{A}^m$ affine varieties, and let $y \in Y$ be arbitrary and $x \in \phi^{-1}(y)$.

Let $\pi_1, \dots, \pi_m$ be the coordinate projections $X \longrightarrow \mathbb{k}$; $(a_1, \dots, a_n) \mapsto a_i$.

Let $i \in \{1, \dots, n\}$ be arbitrary. $\pi_i \in \mathbb{k}[X]$, $\phi$ is a finite map, so $\pi_i$ is integral over $\mathbb{k}[Y]$. $\Rightarrow \exists f_0, f_1, \dots, f_{m-1} \in \mathbb{k}[Y]$ s.t.

$\pi_i^{m_i} + \phi^*(f_{m_i-1})\pi_i^{m_i-1} + \dots + \phi^*(f_i)\pi_i + \phi^*(f_0) = 0$.

We evaluate this in $x$:

$$\begin{aligned}0 &= \pi_i(x)^{m_i} + \phi^*(f_{m_i-1})(x)\pi_i(x)^{m_i-1} + \dots + \phi^*(f_1)(x)\pi_i(x) + \phi^*(f_0)(x) \\ &= \pi_i(x)^{m_i} + f_{m_i-1}(\phi(x))\pi_i(x)^{m_i-1} + \dots + f_1(\phi(x))\pi_i(x) + f_0(\phi(x)) \\ &= \pi_i(x)^{m_i} + f_{m_i-1}(y)\pi_i(x)^{m_i-1} + \dots + f_1(y)\pi_i(x) + f_0(y)\end{aligned}$$

$\Rightarrow \pi_i(x)$ is a root of a polynomial of degree $m_i$ $\Rightarrow$ finitely many possibilities. This holds for each $i$ $\Rightarrow$ we have only finitely many possibilities for $x$.

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