II PROJECTIVE VARIETIES

1. Projective space and projective varieties

October 17, 2025

Let V be a finite-dimensional vector space over $\mathbb{k}$ (still alg. closed). On $V \setminus \{0\}$ we define a relation $u \sim v \Leftrightarrow \exists \lambda \in \mathbb{k} \setminus \{0\}$ such that $v = \lambda u$. This is a equivalence relation. The quotient set $V/_\sim$ is denoted by $\mathbb{P}V$ and called the projective space associated to V. Its elements are equivalence classes in V, so lines through the origin. Dimension of $\mathbb{P}V := \dim V - 1$.

The most common situation is when $V = \mathbb{k}^{n+1}$ for some $n \in \mathbb{N}$. In this case $\dim V = n+1$, so $\dim \mathbb{P}V = n$. Instead of $\mathbb{P}\mathbb{k}^{n+1}$ we write $\mathbb{P}^n_{\mathbb{k}}$ or more usually $\mathbb{P}^n$. We call $\mathbb{P}^n$ the n-dimensional projective space. Its elements (which are lines through 0) are called projective points.

- $\mathbb{P}^1$ is projective line
- $\mathbb{P}^2$ is projective plane

When we work in $\mathbb{P}^n$, we index coordinates in $\mathbb{k}^{n+1}$ from 0 to n: $(x_0, x_1, \dots, x_n)$.
The equivalence class of the point $(x_0, x_1, \dots, x_n)$ is denoted by $(x_0 : x_1 : \dots : x_n)$. In literature there is also notation $[x_0 : x_1 : \dots : x_n]$ or $[x_0, x_1, \dots, x_n]$.
So $(x_0 : x_1 : \dots : x_n)$ is the line in $\mathbb{k}^{n+1}$ through $(x_0, x_1, \dots, x_n)$ and the origin; $x_0, x_1, \dots, x_n$ are called homogeneous coordinates of the point $(x_0 : x_1 : \dots : x_n) \in \mathbb{P}^n$. $x_0, \dots, x_n$ are not all zero.

The points $(x_0 : x_1 : \dots : x_n)$ and $(y_0 : y_1 : \dots : y_n)$ are equal $\Leftrightarrow$ $\exists \lambda \in \mathbb{k} \setminus \{0\}$ s.t. $y_i = \lambda x_i$ $\forall i = 0, 1, \dots, n$.

We can embed $\mathbb{A}^n$ into $\mathbb{P}^n$. For each $i = 0, 1, \dots, n$, we define $U_i = \{(x_0 : x_1 : \dots : x_n) \in \mathbb{P}^n \; ; \; x_i \neq 0\}$.
If $(x_0 : x_1 : \dots : x_n) = (y_0 : y_1 : \dots : y_n)$, then $y_j = \lambda x_j$ $\forall j = 0, 1, \dots, n$ and $\lambda \neq 0$, so $x_i \neq 0 \Leftrightarrow y_i \neq 0$ $\Rightarrow$ the sets U_i are well defined.
We define a map:

$$\begin{aligned} \mathbb{A}^n &\longrightarrow U_i \\ (x_1, \dots, x_n) &\longmapsto (x_1 : \dots : x_i : 1 : x_{i+1} : \dots : x_n). \end{aligned}$$

This is a bijection with the inverse

$$\begin{aligned} U_i &\longrightarrow \mathbb{A}^n \\ (x_0 : x_1 : \dots : x_n) &\longmapsto \left(\tfrac{x_0}{x_i}, \tfrac{x_1}{x_i}, \dots, \tfrac{x_{i-1}}{x_i}, \tfrac{x_{i+1}}{x_i}, \dots, \tfrac{x_n}{x_i}\right). \end{aligned}$$

This map is well defined, because $\frac{\lambda x_j}{\lambda x_i} = \frac{x_j}{x_i}$.

$\Rightarrow$ We can identify $\mathbb{A}^n$ with U_i, (most commonly with U_0) and consider it as a subscape of $\mathbb{P}^n$.

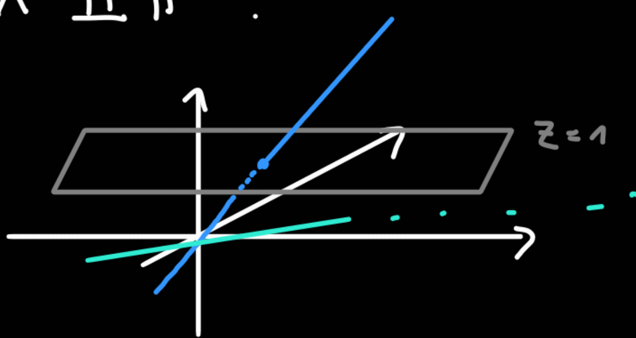
$U_0, U_1, \dots, U_n$ are usually called affine charts of $\mathbb{P}^n$.

$\mathbb{P}^n \setminus U_i$ consists of all points $(x_0 : x_1 : \dots : x_n)$ with $x_i = 0$.
But $x_0, x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n$ are not all zero, so we have

$$\begin{aligned} \mathbb{P}^n \setminus U_i &\longrightarrow \mathbb{P}^{n-1} \\ (x_0 : \dots : x_{i-1} : 0 : x_{i+1} : \dots : x_n) &\longmapsto (x_0 : \dots : x_{i-1} : x_{i+1} : \dots : x_n) \end{aligned}$$

This is a bijection.
We have $\mathbb{P}^n = \mathbb{A}^n \underbrace{\amalg}_{\text{disjoint union}} \mathbb{P}^{n-1}$.

<u>Example:</u> $\mathbb{P}^2$



Lines in xy-plane are in bijection with points in $\mathbb{P}^1$.

$\mathbb{A}^n \amalg \mathbb{P}^{n-1}$ → This is usually called the hyperplane at infinity.

We want to study zero loci of polynomials in $\mathbb{P}^n$.
In $\mathbb{P}^n$ we have homogeneous coordinates:

$$(x_0 : x_1 : \cdots : x_n) = (\lambda x_0 : \lambda x_1 : \cdots : \lambda x_n) \text{ for } \lambda \neq 0$$

so $f(x_0, x_1, \ldots, x_n)$ is not well defined. We restrict to homogeneous polynomials.

<u>Definition</u>: A polynomial $f \in \mathbb{k}[x_0, x_1, \ldots, x_n]$ is homogeneous of degree d if

$$f(\lambda x_0, \lambda x_1, \ldots, \lambda x_n) = \lambda^d f(x_0, x_1, \ldots, x_n)$$

for each $\lambda \in \mathbb{k} \setminus \{0\}$.

Since $\mathbb{k}$ is infinite, this is equivalent to that all monomials of f are of degree d.

<u>Example</u>: $x_0^2 x_1 + x_2^3 - x_4 x_5 x_6$ is homogeneous of degree 3

$\underbrace{x_0^2 x_1}_{\deg 3} + \underbrace{x_2^2}_{\deg 2}$ is not homogeneous

<u>Definition</u>: Let $S \subseteq \mathbb{k}[x_0, x_1, \ldots, x_n]$ be a set of homogeneous polynomials. The set

$$V(S) := \{ (x_0 : x_1 : \cdots : x_n) \in \mathbb{P}^n \mid f(x_0, \ldots, x_n) = 0 \text{ for all } f \in S \}$$

is called the projective zero locus of S. A set $X \subseteq \mathbb{P}^n$ is a projective variety if $X = V(S)$ for some set S of homogeneous polynomials.

If $S = \{f_1, \ldots, f_m\}$ we write $V(f_1, \ldots, f_m)$ instead of $V(\{f_1, \ldots, f_m\})$.

Projective zero loci are well defined:
If $F \in S$ is homogeneous of degree d, then
$F(\lambda x_0, \lambda x_1, \dots, \lambda x_n) = \lambda^d f(x_0, \dots, x_n)$ for all $\lambda \in \mathbb{k} \setminus \{0\}$.
so we get $f(\lambda x_0, \dots, \lambda x_n) = 0 \Leftrightarrow f(x_0, \dots, x_n) = 0$.

Remark: $V(S)$ can mean affine zero locus or projective zero locus. When there can be confusion, we will write $V_a(S)$ or $V_p(S)$.

Examples of projective varieties

(1) $\mathbb{P}^n = V(0)$

(2) $\emptyset = V(1)$, but also $\emptyset = V(x_0, x_1, \dots, x_n)$.

(3) If V is a vector subspace of $\mathbb{k}^{n+1}$, then $\mathbb{P}V$ is a projective variety in $\mathbb{P}^n$, because vector subspaces are defined by homogeneous linear equations. $\mathbb{P}V$ is called the linear subspace of $\mathbb{P}^n$.

(4) Each point is a projective variety: If $a = (a_0 : a_1 : \dots : a_n)$, then $V(a_i x_j - a_j x_i \mid 0 \leq i, j \leq n) = \{a\}$. Proof: HW

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We may define varieties also in product of affine and projective spaces. For example, a variety in $\mathbb{A}^m \times \mathbb{P}^n$ is a zero locus of a set of polynomials in $\mathbb{k}[x_1, \dots, x_m, y_0, \dots, y_n]$ that are homogeneous in the variables $y_0, \dots, y_n$.
Example: $V(x_1^2 y_0^3 - x_2 y_1 y_2^2) \subseteq \mathbb{A}^2 \times \mathbb{P}^2$.
A variety in $\mathbb{P}^m \times \mathbb{P}^n$ is a zero locus of a set of polynomials in $\mathbb{k}[x_0, x_1, \dots, x_m, y_0, \dots, y_n]$ that are homogeneous in $x_0, \dots, x_m$ and (maybe of a different degree) in $y_0, \dots, y_n$.
Example: $V(x_0^2 x_1 y_2^2 - x_2^3 y_0 y_1) \subseteq \mathbb{P}^2 \times \mathbb{P}^2$

$\mathbb{A}^m \times \mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_s}$ similarly

2. Connection between affine and projective varieties

Definition: (1) An affine variety $X \subseteq \mathbb{A}^{n+1}$ is a cone (stožec) if $0 \in X$ and for each $a \in X$ and each $\lambda \in \mathbb{k}$ we have $\lambda a \in X$.

(2) If $X \subseteq \mathbb{A}^{n+1}$ is a cone then the projectiviation (projektivizacija) of X is defined by

$$\mathbb{P}X := \{(a_0 : \cdots : a_n) \in \mathbb{P}^n \mid (a_0, a_1, \ldots, a_n) \in X \setminus \{0\}\} \subseteq \mathbb{P}^n.$$

(3) If $X \subseteq \mathbb{P}^n$ is a projective variety, then the cone over X is defined by $C(X) := \{0\} \cup \{(a_0, \ldots, a_n) \in \mathbb{A}^{n+1} \mid (a_0 : \cdots : a_n) \in X\} \subseteq \mathbb{A}^{n+1}$.

If X is a cone and $(a_0, a_1, \ldots, a_n) \in X$ and $\lambda \in \mathbb{k}$. Since X is a cone, we have $(\lambda a_0, \lambda a_1, \ldots, \lambda a_n) \in X \Rightarrow$ $\mathbb{P}X$ is well-defined.

Proposition: If $X \subseteq \mathbb{A}^{n+1}$ is a cone, then $\mathbb{P}X$ is a projective variety.

Proof: X is an affine variety, so $X = V_a(S)$ for some $S \subseteq \mathbb{k}[x_0, \ldots, x_n]$. We can assume that $S = I_a(X)$. Let $f \in I_a(X)$ is arbitrary. Write $f = \sum_{i=0}^{d} f_i$, where each f_i is homogeneous of degree i. Take arbitrary $a \in X$ and arbitrary $\lambda \in \mathbb{k}$. Since X is a cone, we have $\lambda a \in X$, so $f(\lambda a) = 0$.

$$0 = f(\lambda a) = \sum_{i=0}^{d} f_i(\lambda a) = \sum_{i=0}^{d} \lambda^i f_i(a)$$

This holds for each $\lambda \in \mathbb{k}$, $\mathbb{k}$ infinite $\Rightarrow f_i(a) = 0 \ \forall i$
This holds for each $a \in X$, so $f_i \in I_a(X)$ for each i.

We showed that $I_a(X)$ can be generated by homogeneous polynomials. Let $I_a(X) = (S')$ where S' is a set of homogeneous polynomials. Then

$$\mathbb{P}X = \{(a_0 : \cdots : a_n) \in \mathbb{P}^n \mid (a_0, \ldots, a_n) \in \overset{V(S')}{\overset{\shortparallel}{X}} \setminus \{0\}\}$$

$$= \{(a_0 : \cdots a_n) \in \mathbb{P}^n \mid f(a_0, \ldots, a_n) = 0 \quad \forall f \in S'\}$$

$$\underset{\substack{\uparrow \\ S' \text{ is a set of homogeneous polynomials}}}{=} V_p(S')$$

∎

We proved two more things:

<u>Corollary</u>: If S is a set of homogeneous polynomials then $\mathbb{P}V_a(S) = V_p(S)$.

<u>Corollary</u>: X cone $\Rightarrow I_a(X)$ generated by hom. polynomials.

<u>Proposition</u>: The cone over a projective variety is a cone.

<u>Proof</u>: If $X \neq \emptyset$, then $C(X) = \{0\}$ which is a cone. Assume that $X \subseteq \mathbb{P}^n$ is a non-empty projective variety. Then $X = V_p(S)$ for some set S of non-constant polynomials. If $f \in S$ is hom. of degree d, then $f(\lambda a) = \lambda^d f(a)$ $\forall \lambda \in \mathbb{k}$ and $\forall a \in X$. $\Rightarrow f(0) = 0$.

$$\begin{aligned} C(X) &= \{0\} \cup \{(a_0, \ldots, a_n) \in \mathbb{A}^{n+1} \mid (a_0 : \cdots : a_n) \in X\} \\ &= \{(a_0, a_1, \ldots, a_n) \in \mathbb{A}^{n+1} \mid f(a_0, a_1, \ldots, a_n) = 0 \quad \forall f \in S\} \\ &= V_a(S) \end{aligned}$$

We know $0 \in C(X)$ and that if $a \in V_a(S)$, $\lambda \in \mathbb{k}$, then $f(\lambda a) = \lambda^d f(a) = 0$, so $\lambda a = V_a(S)$. ∎

We proved also:

Corollary: If S is a set of homogeneous polynomials then $C V_p(X) = V_a(S)$.

Corollary: The maps

$$X \longmapsto \mathbb{P}X$$
$$C(X) \longleftarrow X$$

give bijective correspondence between the cones in $\mathbb{A}^{n+1}$ and projective varieties in $\mathbb{P}^n$.

Proof: Both corollaries tell that $\mathbb{P}X$ is a projective variety if X is a cone and that $C(X)$ is a cone if X is a projective variety. We have to prove bijectivity.

Suppose $X \subseteq \mathbb{A}^{n+1}$ is a cone. Then we know from one of the corollaries above that $X = V_a(S)$ for a set of hom. poly. $S \subseteq \mathbb{k}[x_0, \dots, x_n]$. By the other two corollaries:

$$C(\mathbb{P}X) = C(\mathbb{P}V_a(S)) = C(V_p(S)) = V_a(S) = X.$$

Similarly $\mathbb{P}(C(X)) = X$ if X is a projective variety. $\square$

Corollary: Each projective variety is a zero locus of a finite set of homogeneous polynomials.

Proof: $X = V_p(S)$ for some set S of hom. polynomials. Then $C(X) = V_a(X)$. Let $J = I_a(C(X))$. Then we know that J is finitely generated ($\mathbb{k}[x_0, \dots, x_n]$ noetherian) $J = (f_1, \dots, f_n)$. We proved before that homogeneous parts lie in J, and they obviously generate J. So we have a finite set S' of hom. polynomials that generate J.

$\Rightarrow X = \mathbb{P}(C(X)) = \mathbb{P}(V_a(J)) = \mathbb{P}(V_a(S')) = V_p(S')$ ▨

As in the affine case, we can use this corollary to prove:

Lemma: (1) If $\{S_j\}_{j\in J} \subseteq \mathbb{k}[x_0,\dots,x_n]$ is a family of sets of homogeneous polynomials, then

$$V_p\Big(\bigcup_{j\in J} S_j\Big) = \bigcap_{j\in J} V_p(S_j).$$

(2) If $f_1,\dots,f_s, g_1,\dots,g_t \in \mathbb{k}[x_0,\dots,x_n]$ are homogeneous polynomials, then

$$V_p(f_1,\dots,f_s) \cup V_p(g_1,\dots,g_t) = V_p\left(f_i g_j \,;\, \begin{smallmatrix} 1\le i\le s \\ 1\le j\le t\end{smallmatrix}\right).$$

Corollary: (1) $\emptyset$ and $\mathbb{P}^n$ are projective varieties.
(2) The intersection of any family of projective varieties is a projective variety.
(3) The union of finitely many projective varieties is a projective variety.

Projective varieties are therefore exactly the closed sets on some topology on $\mathbb{P}^n$ – Zariski topology on $\mathbb{P}^n$.
As in the affine case, the Zariski topology on subsets of $\mathbb{P}^n$ is the relative topology:
Let $X \subseteq \mathbb{P}^n$. A set $Z \subseteq X$ is Zariski-closed if there exists a proj. var. $Y \subseteq \mathbb{P}^n$ s.t. $Z = X \cap Y$. If X is a proj. var., then it's closed subsets are exactly the subvarieties.

As in the affine case, we also define distinguished open subsets: $D(f) = \mathbb{P}^n \setminus V_p(f)$ where f is a hom. poly.

In a similar way we can define the Zariski topology in any product $\mathbb{A}^m \times \mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_s}$.

Definition: Let $X \subseteq \mathbb{P}^n$. X is reducible if $X = X_1 \cup X_2$ for some proper closed subsets $X_1, X_2 \subsetneq X$, irreducible otherwise.

If X is a proj. var., the X is reducible $\Leftrightarrow$ X is a union of two proper subvarieties.

Theorem: Each proj. var. $X \subseteq \mathbb{P}^n$ can be written as $X = X_1 \cup X_2 \cup \dots \cup X_m$ where $m \in \mathbb{N}_0$ and $X_1, \dots, X_m$ are irreducible proj. var. Moreover, if $X_i \not\subseteq X_j$ whenever $i \neq j$, then this decomposition is unique up to an order. In this case $X_1, \dots, X_m$ are called irreducible components of X.

$\mathbb{P}^n = \mathbb{A}^n \amalg \mathbb{P}^{n-1}$

Recall: $U_i = \{ (x_0 : \dots : x_n) \in \mathbb{P}^n ; x_i \neq 0 \} = D(x_i)$
We identified U_i with $\mathbb{A}^n$.
$U_i = D(x_i)$ are open in Zariski topology, moreover, $\{U_0, \dots, U_n\}$ is an open cover of $\mathbb{P}^n$.
$\Rightarrow$ We can consider $\mathbb{A}^n$ as an open subset of $\mathbb{P}^n$.
We have 2 Zariski topologies on $\mathbb{A}^n$: one defined by affine varieties and one as a relative topology in $\mathbb{P}^n$.
Are they equal?

$\boxed{\text{We will identify } \mathbb{A}^n \text{ with } V_0.}$

<u>Definition</u>: Let $f \in \Bbbk[x_0, \dots, x_n]$ be a homogeneous polynomial. Dehomogenization of f is the polynomial

$$f^{(d)} := f(1, x_1, \dots, x_n) \in \Bbbk[x_1, \dots, x_n].$$

Dehomogenization is evaluation $x_0 = 1$, so it is a ring homomorphism, so:

$$(fg)^{(d)} = f^{(d)} \cdot g^{(d)},$$
$$(f+g)^{(d)} = f^{(d)} + g^{(d)}.$$

<u>Definition</u>: Let $f \in \Bbbk[x_1, \dots, x_n]$ be a non-zero polynomial of degree d. Then

$$f^{(h)} := x_0^d f\left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right)$$

is a polynomial, called the homogenization of f.

<u>Example</u>: $f(x_1, x_2, x_3) = x_1^3 - x_2 x_3 + 2x_2^4$

$\deg f = 4 \Rightarrow f^{(h)}(x_0, x_1, x_2, x_3) = x_0 x_1^3 - x_0^2 x_2 x_3 + 2x_2^4$

We have $(fg)^{(h)} = f^{(h)} g^{(h)}$, but $(f+g)^{(h)} \neq \underbrace{f^{(h)} + g^{(h)}}_{\text{this polynomial may not be homogeneous}}$

<u>Proposition</u>: Each affine variety $X \subseteq \mathbb{A}^n \equiv V_0 \subseteq \mathbb{P}^n$ is of the form $X = Z \cap U_0$ for some projective variety Z. More precisely, if $X = V_a(f_1, \dots, f_m)$, we may take $Z = V_p(f_1^{(h)}, \dots, f_m^{(h)})$.

<u>Proof</u>: Let f_i be of degree d_i for each i.

$$Z \cap U_0 = V_p(f_1^{(h)}, \dots, f_m^{(h)}) \cap U_0$$

$$= \{ (a_0 : a_1 : \dots : a_n) \in \mathbb{P}^n \mid a_0 \neq 0 \ \forall i: f_i^{(h)}(a_0, \dots, a_n) = 0 \}$$

$$= \{ (1 : \tfrac{a_1}{a_0} : \dots : \tfrac{a_n}{a_0}) \in \mathbb{P}^n \mid a_0 \neq 0, \forall i: f_i^{(h)}(1, \tfrac{a_1}{a_0}, \dots, \tfrac{a_n}{a_0}) = 0 \}$$

$$= \{ (1 : \tfrac{a_1}{a_0} : \dots : \tfrac{a_n}{a_0}) \in \mathbb{P} \mid a_0 \neq 0, \forall i: \underbrace{a_0^{d_i}}_{\neq 0} f_i(1, \tfrac{a_1}{a_0}, \dots, \tfrac{a_n}{a_0}) = 0 \}$$

$$= \{ (1 : \tfrac{a_1}{a_0} : \dots : \tfrac{a_n}{a_0}) \in \mathbb{P} \mid a_0 \neq 0, \forall i: f_i(1, \tfrac{a_1}{a_0}, \dots, \tfrac{a_n}{a_0}) = 0 \}$$

$$\cong \{ (b_1, \dots, b_n) \in \mathbb{A}^n \mid \forall i.\ f_i(b_1, \dots, b_n) = 0 \}$$

$$= V_a(f_1, \dots, f_n) = X$$

∎

Corollary: Both Zariski topologies on $\mathbb{A}^n$ coincide.

We often study open subsets of projective varieties. Such sets are called *quasiprojective* varieties. Important examples of quasiprojective varieties are proj. varieties and affine varieties.

Definition: Let $X \subseteq \mathbb{A}^n \subseteq \mathbb{P}^n$ be an affine variety. The *projective closure* of X is the smallest projective variety that contains X. Notation: $\overline{X}$

In general $\overline{X} \neq V_p(f_1^{(h)}, \dots, f_m^{(h)})$ if $X = V_a(f_1, \dots, f_m)$.

Example: $X = V_a(x_1, x_2 - x_1^2) = \{(0,0)\}$. $\overline{X}$ also has to be one point: $\overline{X} = \{[1:0:0]\}$, but
$V_p(x_1, x_0x_2 - x_1^2) = \{[1:0:0], [0:0:1]\} = \overline{X}$.

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Proposition: Let $X \subseteq \overset{= U_0 \subseteq \mathbb{P}^n}{\mathbb{A}^n}$ be a an affine variety and $\overline{X} \subseteq \mathbb{P}^n$ its projective closure. Then:

(1) $\overline{X} \cap U_0 = X$ (2) X irreducible $\Rightarrow$ $\overline{X}$ irreducible

(3) No irreducible component of $\overline{X}$ lies in $V_p(x_0)$ (= hyperplane at infinity).

Proof: (1) We know that $X = Z \cap V_0$ for some projective variety Z. Z is a projective variety that contains X, so it also contains $\bar{X}$.

(2) Suppose that $\bar{X} = Z_1 \cup Z_2$ for some projective varieties Z_1, Z_2. By (1) we get

$$X = \bar{X} \cap U_0 = (Z_1 \cap U_0) \cup (Z_2 V_0)$$

$Z_1 \cap U_0$ and $Z_2 \cap U_0$ are affine varieties, so irreducibility of X implies $X = Z_i \cap V_0$ for some $i = 1,2$. Z_i is a projective variety that contains X, so $\bar{X} \subseteq Z_i \Rightarrow \bar{X}$ irreducible.

(3) Let $\bar{X} = Z_1 \cup \cdots \cup Z_m$ be the decomposition into irreducible components. Suppose that $Z_1 \subseteq V_p(x_0)$.

$$\begin{aligned} X = \bar{X} \cap U_0 &= (Z_1 \cup \cdots \cup Z_m) \cap U_0 \\ &= \underbrace{(Z_1 \cap V_0)}_{= \emptyset} \cup ((Z_2 \cup \cdots \cup Z_1) \cap V_0) \\ &= (Z_2 \cup \cdots \cup Z_m) \cap U_0 \end{aligned}$$

$Z_2 \cup \cdots \cup Z_m$ is a projective variety that contains X, so it contains $\bar{X}$ and therefore

$$Z_1 \subseteq Z_2 \subseteq \cdots \subseteq Z_m$$

Z_1 is irreducible, therefore $Z_1 \subseteq Z_i$ for some $i = 2, \dots, m$. This contradicts the fact that Z_i are components. $\square$

3. Projective algebra-geometry correspondence

<u>Definition</u>: A ring/algebra R is graded if we can write it as a direct sum of abelian groups/$\mathbb{k}$-vector spaces $R = \bigoplus_{d=0}^{\infty} R_d$ such that $R_d \cdot R_e \subseteq R_{d+e}$ for all $d, e \in \mathbb{N}_0$.

slovene: stopničast kolobar

If $f \in R_d$ and $g \in R_e$, then $fg \in R_{d+e}$.

We say that elements of $R_d \setminus \{0\}$ are homogeneous of degree d.

<u>Example</u>: $R = \mathbb{k}[x_0, x_1, \dots, x_n]$ is a graded $\mathbb{k}$-algebra:
$R = \bigoplus_{d=0}^{\infty} R_d$, where $R_d = \{0\} \cup \{\text{homogeneous polynomials of degree } d\}$.

Let $f \in R$. Then f can be uniquelly decomposed as $f = \sum_{d=0}^{\infty} f_d$ where $f_d \in R_d$ for each d and only finitely many f_d's are nonzero. The decomposition $f = \sum_{d=0}^{\infty} f_d$ is called the homogeneous decomposition of f.
If $f \neq 0$, then the degree of f is the largest d s.t. $f_d \neq 0$.

<u>Definition</u>: Let R be a graded ring/algebra. An ideal $I \trianglelefteq R$ is homogeneous if it can be generated by homogeneous elements.

<u>Example</u>: $I = (x_1, x_2 - x_1^2) = I(x_1, x_2)$ is a homogeneous ideal.

<u>Example</u>: If X is a cone, then we showed that $I_a(X)$ is homogeneous.

Proposition: Let $R = \bigoplus_{d=0}^{\infty} R_d$ be a graded ring and $J, J_1, J_2 \trianglelefteq R$. Then the following holds:

(1) J is homogeneous $\iff$ for each $f \in J$ with homogeneous decomposition $f = \sum_{d=0}^{\infty} f_d$ we have $f_d \in J$ for each d.

(2) J, J_1, J_2 homogeneous $\Rightarrow$ $J_1 + J_2$, $J_1 \cap J_2$, $J_1 J_2$, $\sqrt{J}$ homogeneous

(3) J homogeneous, then R/J is a graded ring with the homogeneous decomposition:

$$R/J = \bigoplus_{d=0}^{\infty} R_d/(R_d \cap J).$$

$$\cong \text{ isomorphism theorem}$$

$$(R_d + J)/J$$

(4) If R is noetherian and J is a homogeneous ideal, then J can be generated by finitely many homogeneous elements.

Proof: Exercise.

Definition: (1) For a homogeneous ideal $J \trianglelefteq \Bbbk[x_0, x_1, \dots, x_n]$ we define:

$$V(J) = V_p(J) := \{x \in \mathbb{P}^n \mid f(x) \ \forall \text{ homogeneous } f \in J\}.$$

(2) For $X \subseteq \mathbb{P}^n$ we define the ideal of X as

$$I(X) = I_p(X) := \{f \in \Bbbk[x_0, \dots, x_n] \text{ homogeneous} \mid f(x) = 0 \ \forall x \in X\}.$$

Remark: (1) $V(J)$ is well defined, because it is defined only using homogeneous polynomials. If S is a homogeneou set of generators of J, then $V_p(J) = V_p(S)$. $V_p(J)$ is a projective variety.

(2) The set of all homogeneous polynomials vanishing on X is not an ideal. To get $I_p(X)$, we must take the ideal generated by them.

Lemma: (1) If $X \subseteq Y \subseteq \mathbb{P}^n$, then $I_p(Y) \subseteq I_p(X)$.

(2) For homogeneous ideals $I_1 \subseteq I_2$ we have $V_p(I_2) \subseteq V_p(I_1)$.

October 27, 2025

Proposition: Let $X \subseteq \mathbb{P}^n$ be an non-empty projective variety. Then $I_p(X) = I_a(C(X))$.

Proof: $(\subseteq)$: Let $f \in I_p(X)$ and $f = \sum_{d=0}^{\infty} f_d$ be the homogeneous decomposition of f.

We know that $I_p(X)$ is a homogeneous ideal, so $f_d \in I_p(X)$ for each d.

$\Rightarrow f_d(x_0, x_1, \dots, x_n) = 0 \quad \forall (x_0, \dots, x_n) \in X$

$\Rightarrow f_d(x_0, \dots, x_n) = 0$ for each point in the cone

$\Rightarrow f_d \in I_a(C(X))$ for each $d \Rightarrow f \in I_a(C(X))$.

$(\supseteq)$: We proved that $I_a(C(X))$ is a homogeneous ideal, so it is generated by homogeneous elements.

$\Rightarrow$ Let $I_a(C(X)) = (g_j \mid j \in J)$. $g_j \in I_a(C(X))$

$\Rightarrow g_j(x_0, x_1, \dots, x_n) = 0 \quad \forall (x_0, x_1, \dots, x_n) \in C(X)$

$\Rightarrow g_j(x_0 : x_1 : \dots : x_n) = 0 \quad \forall (x_0 : x_1 : \dots : x_n) \in X$

$\Rightarrow g_j \in I_p(X) \; \forall j \in J \Rightarrow I_a(C(X)) \subseteq I_p(X)$ $\square$

Corollary: $I_p(X)$ is always a radical ideal.

Theorem: If X is a projective variety, then $V_p(I_p(X)) = X$.

Proof: If $X = \emptyset$, then $I_p(X) = \mathbb{k}[x_0, x_1, \dots, x_n]$ and $V_p(I_p(X)) = \emptyset$.

If $X \neq \emptyset$, then

$$V_p(I_p(X)) = \mathbb{P}V_a(I_p(X)) \overset{\text{previous proposition}}{=} \mathbb{P}V_a(I_a(C(X)))$$

$$= \mathbb{P}(C(X)) = X.$$ $\square$

As in the affine case, we have:

<u>Proposition</u>: If $X \subseteq \mathbb{P}^n$ is any set, then $\bar{X} = V_p(I_p(X))$.

Affine weak Nullstelensatz: If $J \neq (1)$ is a proper ideal, then $V_a(J) \neq \emptyset$.

In projective case: $V_p(x_0, x_1, \dots, x_n) = \emptyset$

<u>Definition</u>: The ideal $I_0 = (x_0, x_1, \dots, x_n) \trianglelefteq \Bbbk[x_0, x_1, \dots, x_n]$ is called the irrelevant ideal.

<u>Theorem [projective weak Nullstellensatz]</u>: For a proper homogeneous ideal $J \trianglelefteq \Bbbk[x_0, x_1, \dots, x_n]$ we have $V_p(J) = \emptyset \Leftrightarrow \sqrt{J} = I_0$.

<u>Proof</u>: ($\Leftarrow$): Suppose $\sqrt{J} = I_0 = (x_0, x_1, \dots, x_n)$.

$\Rightarrow x_i \in \sqrt{J}$ for each $i \Rightarrow \forall i.\ \exists N_i$ s.t. $x_i^{N_i} \in J$.

Suppose that $(a_0 : a_1 : \dots : a_n) \in V_p(J)$.

$\Rightarrow \forall i.\ a_i^{N_i} = 0 \Rightarrow a_i = 0\ \forall i \Rightarrow \underbrace{(0 : 0 : \dots : 0)}_{\text{(not a projective point)}} \in V_p(J) \Rightarrow V_p(J) = \emptyset$

($\Rightarrow$): Suppose that $V_p(J) = \emptyset$. $J \neq (1) \Rightarrow V_a(J) \neq \emptyset$

J is a homogeneous ideal, so $J = (S)$ where S is a of homogeneous polynomials. We know $V_p(J) = V_p(S)$, $V_a(J) = V_a(S)$, $V_p(S) = \mathbb{P} V_a(S) \Rightarrow \underset{\emptyset}{\underset{\parallel}{V_p(J)}} = \mathbb{P}\, \underbrace{V_a(J)}_{\text{non-empty cone}}$

The only possibility is $V_a(J) = \{(0, 0, \dots, 0)\}$.

$\Rightarrow I_a(V_a(J)) = \sqrt{J} = (x_0, x_1, \dots, x_n)$. $\square$

<u>Theorem [projective Nullstellensatz]</u>: For a homogeneous ideal $J \trianglelefteq \Bbbk[x_0, x_1, \dots, x_n]$ with $\sqrt{J} \neq I_0$ we have $I_p(V_p(J)) = \sqrt{J}$.

<u>Proof</u>: If $V_p(J) = \emptyset$, then $J = (1)$ by the projective weak Nullstellensatz.

Then $\sqrt{J} = (1) = I_p(V_p(J)) = I_p(\emptyset)$.

Assume now that $V_p(J) \neq \emptyset$. J is a homogeneous ideal, so $J=(S)$ for some set S of homogeneous polynomials.

$$I_p(V_p(J)) = I_p(V_p(S)) = I_a(C(V_p(S))) = I_a(C(\mathbb{P}(V_a(S))))$$
$$= I_a(V_a(S)) = I_a(V_a(J)) = \underset{\uparrow \text{ affine Nullstelensatz}}{\sqrt{J}} \qquad ▨$$

The maps I_p and V_p have similar properties as I_a and V_a. In some cases we need to assume that some ideal is not irrelevant.

Corollary: (1) We have a bijection:

$$\left\{\begin{array}{c}\text{projective}\\ \text{varieties}\\ \text{in } \mathbb{P}^n\end{array}\right\} \underset{V_p}{\overset{I_p}{\rightleftarrows}} \left\{\begin{array}{l}\text{homogeneous radical ideals}\\ \text{in } \mathbb{k}[x_0,\dots,x_n] \text{ different from}\\ I_0=(x_0,\dots,x_n)\end{array}\right\}$$

(2) We have a bijection:

$$\left\{\begin{array}{c}\text{irreducible}\\ \text{projective}\\ \text{varieties}\\ \text{in } \mathbb{P}^n\end{array}\right\} \underset{V_p}{\overset{I_p}{\rightleftarrows}} \left\{\begin{array}{l}\text{homogeneous prime ideals}\\ \text{in } \mathbb{k}[x_0,\dots,x_n] \text{ different}\\ \text{from } I_0=(x_0,\dots,x_n)\end{array}\right\}$$

Definition: Let $J \triangleleft \mathbb{k}[x_1,\dots,x_n]$. The homogenization of J is the ideal generated by the homogenizations of all elements from J:

$$J^{(h)} = (f^{(h)} \mid f \in J) \triangleleft \mathbb{k}[x_0,x_1,\dots,x_n].$$

Proposition: Let $X \subseteq \mathbb{A}^n$ be an affine variety, $\mathbb{A}^n \cong U_0 \subseteq \mathbb{P}^n$. Let $J = I_a(X)^{(h)} \triangleleft \mathbb{k}[x_0,\dots,x_n]$. Then $I_p(X) = J$ and $\underset{\text{Zariski closure}}{\overline{X}} = V_p(J) = V_p(I_a(X)^{(h)})$.

Proof: It is enough to prove that $J = I_p(X)$.

($\subseteq$): It is enough to show that $f^{(h)} \in I_p(X)$ $\forall f \in I_a(X)$. Let $f \in I_a(X)$ be arbitrary. $I_a(X) \triangleleft \mathbb{k}[x_1,\dots,x_n] \subseteq \mathbb{k}[x_0,\dots,x_n]$. Let $a \in X$, $a = (a_0 : a_1 : \dots : a_n)$. Since $X \subseteq U_0$, we may assume that $a_0 = 1$.

$\Rightarrow f(1,a_1,\dots,a_n) = 0$ $(f \in I_a(X), a \in X)$

$$f^{(h)}(a_0,a_1,\dots,a_n) = \underset{=1}{a_0^d} f\Big(\frac{a_1}{\underset{=1}{a_0}},\dots,\frac{a_n}{\underset{=1}{a_0}}\Big) = 0 \Rightarrow f^{(h)} \in I_p(X)$$

($\supseteq$): It is enough to show that $f \in J$ for each homogeneous f that vanishes on X.
Let $f \in I_p(X)$. If $f = x_0 \cdot g$, then also $g \in I_p(X)$, because $X \subseteq U_0$ and $x_0 \neq 0$ on X. So we assume that f is not divisible by x_0. Let $a = (a_0, a_1, \dots, a_n) \in X$ be arbitrary. We can assume $a_0 = 1$.

$f \in I_p(X) \Rightarrow f(1, a_1, \dots, a_n) = 0$

$\Rightarrow$ The polynomial $f(1, x_1, \dots, x_n)$ vanishes on X.

$\Rightarrow f(1, x_1, \dots, x_n) \in I_a(X)$

$f = f(1, x_1, \dots, x_n)^{(h)} \quad \Rightarrow f \in J$ ▨

(not divisible by x_0)

<u>Example</u>: $X = V(x_1, x_2 - x_1^2)$

The ideal $\underset{J}{(x_1, x_2 - x_1^2)} = (x_1, x_2)$ is homogeneous.

$\Rightarrow J^{(h)} = (x_1, x_2) \trianglelefteq \mathbb{k}[x_0, x_1, x_2] \Rightarrow \overline{X} = V_p(x_1, x_2) = \{[1:0:0]\}$.

<u>Corollary</u>: Let $X \subseteq \mathbb{P}^n$ be a projective variety and $J_i = I_a(X \cap U_i)$ for $i = 0, \dots, n$; $J_i \trianglelefteq \mathbb{k}[x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_n]$. Then $I_p(X) = J_0^{(h)} \cap J_1^{(h)} \cap \dots \cap J_n^{(h)}$.

<u>Proof</u>: $X = \bigcup_{i=0}^{n} (X \cap U_i) \quad / I_p$

$$I_p(X) = I_p\left(\bigcup_{i=0}^{n} (X \cap U_i)\right) = \bigcap_{i=0}^{n} I_p(X \cap U_i) \overset{\text{proposition}}{=} \bigcap_{i=0}^{n} I_a(X \cap U_i)^{(h)}.$$ ▨