III. MAPS BETWEEN VARIETIES

1. Polynomial maps and coordinate ring

⚠ In this section everything will be affine.

Definition: Let $X \subseteq \mathbb{A}^n$ be an affine variety and $f \in \mathbb{k}[x_1, \dots, x_n]$ a polynomial. The map $X \longrightarrow \mathbb{k}$, $a \longmapsto f(a)$ is called a polynomial function on X.
The set of all polynomial functions on X is a ring for point-wise addition and multiplication. We call it the coordinate ring of X. Notation: $\mathbb{k}[X]$

Proposition: Let $X \subseteq \mathbb{A}^n$ be an affine variety. Then $\mathbb{k}[X] \cong \mathbb{k}[x_1, \dots, x_n]/I(X)$.

Proof: Let $\Phi : \mathbb{k}[x_1, \dots, x_n] \longrightarrow \mathbb{k}[X]$ be the map defined by $f \longmapsto (a \longmapsto f(a))$. Φ is a ring homomorphism and it is clearly surjective, so $\mathbb{k}[X] \cong \mathbb{k}[x_1, \dots, x_n]/\ker \Phi$.
$f \in \ker \Phi \Leftrightarrow f(a) = 0 \ \forall a \in X \Leftrightarrow f(a) \in I(X)$.
$\Rightarrow \mathbb{k}[X] \cong \mathbb{k}[x_1, \dots, x_n]/I(X)$. ▨

Corollary: $\mathbb{k}[\mathbb{A}^n] = \mathbb{k}[x_1, \dots, x_n]$ $\qquad I(\mathbb{A}^n) = (0)$

Proposition:
(1) $\mathbb{k}[X]$ is without nilpotents. We say it is reducible.
(2) $\mathbb{k}[X]$ is a domain $\Leftrightarrow$ X is irreducible.

Proof: (1) Some power of a function is $0 \Leftrightarrow$ the function is 0.
(2) $\mathbb{k}[X] \cong \mathbb{k}[x_1, \dots, x_n]/I(X)$ is a domain $\Leftrightarrow I(X)$ is a prime ideal
$\Leftrightarrow X$ is irreducible. ▨

Remark: If $X = X_1 \cup \cdots \cup X_m$ is the decomposition of X into irreducible components, then $\Bbbk[X] \cong \Bbbk[X_1] \times \cdots \times \Bbbk[X_m]$.
Commutative algebra: Chinese Remainder Theorem

Definition: Let $X \subseteq \mathbb{A}^n$ be a variety.
(1) A *subvariety* of X is any subset of the form
$$V_X(S) := \{a \in X \mid f(a) = 0 \ \forall f \in S\}$$
where $S \subseteq \Bbbk[X]$.

(2) For any subset $Y \subseteq X$ we define the *ideal of Y* in $\Bbbk[X]$ by $I_X(Y) = \{f \in \Bbbk[X] \mid f(a) = 0 \ \forall a \in Y\} \trianglelefteq \Bbbk[X]$.

The maps I_X and V_X have the following properties:
(1) If $S \subseteq \Bbbk[X]$ and $J \trianglelefteq \Bbbk[X]$ is the ideal generated by S, then $V_X(S) = V_X(J)$.
(2) $\Bbbk[X]$ is a quotient of a noetherian ring, so it is noetherian $\Rightarrow$ subvarieties of X are of the form $V_X(S)$ for finite S.
(3) Subvarieties of X are precisely the varieties that are contained in X.
(4) If Y is a subvariety of X, then
$$\Bbbk[X]/I_X(Y) \cong \left(\Bbbk[x_1,\dots,x_n]/I(X)\right) / \left(I(Y)/I(X)\right) \overset{\text{isomorphism theorem}}{\cong} \Bbbk[x_1,\dots,x_n]/I(Y) \cong \Bbbk[Y]$$
(5) $V_X(I_X(Y)) = Y$ if Y is a subvariety of X.
(6) **Relative Nullstellensatz:**
If $J \trianglelefteq \Bbbk[X]$, then $I_X(V_X(J)) = \sqrt{J}$.
(7) Versions of properties from the proposition with 16 properties for V and I hold.
(8) There is a bijective correspondence between subvarieties of X and radical ideals of $\Bbbk[X]$.

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Definition: Let $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^m$ be affine varieties. A map $\phi : X \longrightarrow Y$ is a **polynomial map** if there exist polynomials $f_1, \dots, f_m \in \mathbb{k}[x_1, \dots, x_n]$ such that $\phi(a) = (f_1(a), \dots, f_m(a))$ for each $a \in X$.

Lemma: Polynomial maps are continuous in the Zariski topology.

Proof: Let $X \subseteq \mathbb{A}^n$, $Y \subseteq \mathbb{A}^m$ and $\phi : X \to Y$ a polynomial map. Then there exist polynomials $f_1, \dots, f_m \in \mathbb{k}[x_1, \dots, x_n]$ s.t. $\phi(a) = (f_1(a), \dots, f_m(a))$ $\forall a \in X$. Let $Z \subseteq Y$ be a closed subset. We have to prove that $f^{-1}(Z)$ is closed. Z is closed in Y, which is closed in $\mathbb{A}^m$ so Z is closed in $\mathbb{A}^m$, so Z is an affine variety. Therefore there exist $g_1, \dots, g_\ell \in \mathbb{k}[x_1, \dots, x_m]$ s.t. $Z = V(g_1, \dots, g_\ell)$.

$$\begin{aligned} \phi^{-1}(Z) &= \{a \in X \mid \phi(a) \in Z\} \\ &= \{a \in X \mid g_i(\phi(a)) = 0 \ \forall i = 1, \dots, \ell\} \\ &= \{a \in X \mid g_i(f_1(a), \dots, f_m(a)) = 0 \ \forall i\} \end{aligned}$$

Observe that $g_i(f_1(x_1, \dots, x_n), \dots, f_m(x_1, \dots, x_n))$ is a polynomial for each i. So $\phi^{-1}(Z)$ is an affine variety, so closed in X. ▨

Corollary: If $\phi : X \to Y$ is a polynomial map and $S \subseteq X$ any subset, then $\phi(\overline{S}) = \overline{\phi(S)}$.

Corollary: If X is an irreducible variety and $\phi : X \longrightarrow Y$ is a polynomial map, then $\overline{\phi(X)}$ is irreducible.

Proof: Assume that $\overline{\phi(X)} = Z_1 \cup Z_2$ for two closed subsets $Z_1, Z_2 \subseteq \overline{\phi(X)}$. If $a \in X$ is arbitrary, then $\phi(a) \in \overline{\phi(X)} = Z_1 \cup Z_2$, so $\phi(a) \in Z_i$ for some $i \in \{1, 2\}$. We showed that $X \subseteq \phi^{-1}(Z_1) \cup \phi^{-1}(Z_2)$.

$a \in \phi^{-1}(Z_i)$

Since ϕ is continuous, $\phi^{-1}(Z_1)$ and $\phi^{-1}(Z_2)$ are closed, and irreducibility of X implies $X \subseteq \phi^{-1}(Z_1)$ for some $i \in \{1,2\}$.

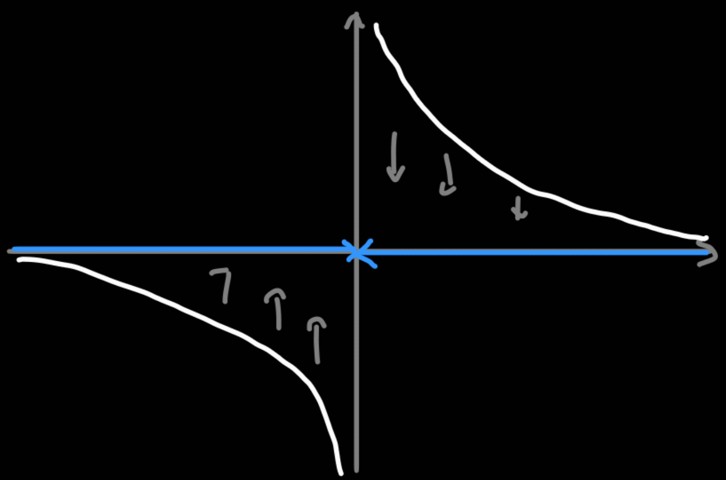
$\Rightarrow \phi(X) \subseteq \phi(\phi^{-1}(Z_i)) \subseteq Z_i$

Z_i is closed, so $\overline{\phi(X)} \subseteq Z_i \Rightarrow Z_i$ is not a proper subset of $\overline{\phi(X)} \Rightarrow \overline{\phi(X)}$ is irreducible $\blacksquare$

Corollary: If $\phi: \mathbb{A}^n \longrightarrow X$ is a polynomial map, then $\overline{\phi(\mathbb{A}^n)}$ is irreducible.

The image of a polynomial map is not necessarily closed.

Example: $X = V(xy-1)$, $\phi: X \longrightarrow \mathbb{A}^1$ projection



$\phi(X) = \mathbb{A}^1 \setminus \{0\}$

The image of a polynomial map is also not necessarily open.

Example: $\phi: \mathbb{A}^2 \longrightarrow \mathbb{A}^2$

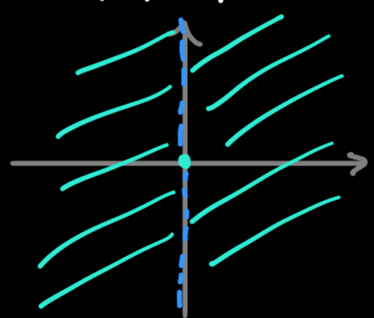
$(x,y) \longmapsto (x, xy)$

What is $\phi(\mathbb{A}^2)$?

If $x \neq 0$, we can get all pairs of the form (x,z), $z \in \mathbb{A}^1$.

If $x = 0$, we get only $(0,0)$.

$\Rightarrow \phi(\mathbb{A}^2) = (\mathbb{A}^1 \setminus \{0\}) \times \mathbb{A}^1 \cup \{(0,0)\}$.



This is not open in $\mathbb{A}^2$.

Definition: Affine varieties X and Y are isomorphic if there exist polynomial maps $\phi: X \longrightarrow Y$ and $\psi: Y \longrightarrow X$ s.t. $\phi \circ \psi = id_Y$ and $\psi \circ \phi = id_X$.

Bijective polynomial maps are not necessarilly isomorphisms.

Example: $X = V(x^2 - y^3)$, $\quad \begin{aligned} \phi: \mathbb{A}^1 &\longrightarrow X \\ t &\longmapsto (t^3, t^2) \end{aligned}$

$\phi(\mathbb{A}^1)$ indeed lies in X: $(t^3)^2 - (t^2)^3 = 0$.

ϕ is a polynomial map and it is injective:

If $(t^3, t^2) = (u^3, u^2)$ and $t \neq 0$, then $\left.\begin{aligned} t^3 = u^3 \\ t^2 = u^2 \end{aligned}\right\} \Rightarrow t = u$

The map ϕ is also surjective: Let $(a,b) \in X$ and suppose $a, b \neq 0$.

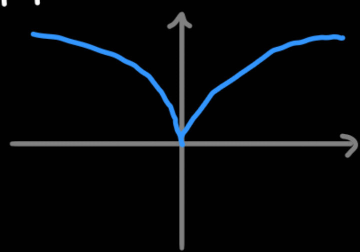
$$\phi\left(\frac{a}{b}\right) = \left(\frac{a^3}{b^3}, \frac{a^2}{b^2}\right) \underset{\uparrow}{=} \left(\frac{a^3}{a^2}, \frac{b^3}{b^2}\right) = (a, b)$$

$(a,b) \in X \Rightarrow a^2 = b^3$

$\Rightarrow \phi$ is a bijective polynomial map, but it is not an isomorphism (proof later).

X has a cusp in $(0,0)$,

but $\mathbb{A}^1$ is smooth



Lemma: Composition of polynomial maps is a polynomial map.

Proof: $\phi: X \to Y$, $\psi: Y \longrightarrow Z$, $X \subseteq \mathbb{A}^n$, $Y \subseteq \mathbb{A}^m$, $Z \subseteq \mathbb{A}^\ell$,

ϕ, ψ polynomial maps.

$\Rightarrow \exists f_1, \dots, f_m \in \mathbb{k}[x_1, \dots, x_n]$ s.t. $\phi(a) = (f_1(a), \dots, f_m(a))$ $\forall a \in X$.

$\exists g_1, \dots, g_\ell \in \mathbb{k}[x_1, \dots, x_n]$ s.t. $\psi(a) = (g_1(a), \dots, g_\ell(a))$ $\forall a \in Y$.

$\forall a \in X$: $(\psi \circ \phi) = \psi(f_1(a), \dots, f_m(a))$

$$= (g_1(f_1(a), \dots, f_m(a)), \dots, g_\ell(f_1(a), \dots, f_m(a)))$$

Components are polynomials, as they are compositions of polynomials. $\blacksquare$

Lemma: Let $X \subseteq \mathbb{A}^n$, $Y \subseteq \mathbb{A}^m$, $\phi: X \to Y$ a map and $\pi_i: Y \to \mathbb{A}^1$ be the projection to the i-th component. ϕ is a polynomial map $\Longleftrightarrow$ all compositions $\pi_i \circ \phi$ are polynomial functions.

Proof: ($\Rightarrow$): Follows from the previous lemma.
($\Leftarrow$): Suppose that $\pi_i \circ \phi$ is a polynomial function for each i. Then $\forall i$ $\exists f_i \in \mathbb{k}[x_1, \dots, x_n]$ s.t. $\pi_i(\phi(a)) = f_i(a)$ $\forall a \in X$.
$\Rightarrow \phi(a) = (f_1(a), \dots, f_m(a)) \Rightarrow \phi$ is a polynomial map. $\blacksquare$

Corollary: Let $\phi: X \to Y$ be a polynomial map and $g \in \mathbb{k}[Y]$. Then $g \circ \phi \in \mathbb{k}[X]$. $X \xrightarrow{\phi} Y \xrightarrow{g} \mathbb{k}$

Definition: Let ϕ and g be as in the corollary. The element $g \circ \phi \in \mathbb{k}[X]$ is called the pullback (slovene: povlek) of g under ϕ. We will denote it by $\phi^*(g)$.

Let $\phi: X \to Y$ be a polynomial map. Then we have a map $\phi^*: \mathbb{k}[Y] \to \mathbb{k}[X]$.

Lemma: ϕ^* is a homomorphism of $\mathbb{k}$-algebras.

Lemma: If $\phi: X \to Y$ and $\tau: Y \to Z$ are polynomial maps, then $(\tau \circ \phi)^* = \phi^* \circ \tau^*$.

Theorem: The map $\phi \longmapsto \phi^*$ gives a bijection between the set of polynomial maps $X \to Y$ and the set of $\mathbb{k}$-algebra homomorphisms $\mathbb{k}[Y] \longrightarrow \mathbb{k}[X]$.

Proof: $\phi^*: \mathbb{k}[Y] \longrightarrow \mathbb{k}[X]$ is an algebra homomorphism by the lemma. We have to prove bijectivity.

Injectivity: Suppose $\phi, \psi : X \to Y$ are polynomial maps s.t. $\phi^* = \psi^* : \mathbb{k}[Y] \to \mathbb{k}[X]$.

$$\phi^*(g) = \psi^*(g) \quad \forall g \in \mathbb{k}[Y]$$
$$g \circ \phi = g \circ \psi \quad \forall g \in \mathbb{k}[Y]$$
$$g(\phi(a)) = g(\psi(a)) \quad \forall g \in \mathbb{k}[Y],\ \forall a \in X$$

Let $f_1, \dots, f_m \in \mathbb{k}[x_1, \dots, x_n]$ s.t. $\phi(a) = (f_1(a), \dots, f_m(a))$ $\forall a \in X$
$h_1, \dots, h_m \in \mathbb{k}[x_1, \dots, x_n]$ s.t. $\psi(a) = (h_1(a), \dots, h_m(a))$ $\forall a \in X$

$\Rightarrow g(f_1(a), \dots, f_m(a)) = g(h_1(a), \dots, h_m(a))$ $\forall g \in \mathbb{k}[Y]$ $\forall a \in X$

For g we take projection to the i-th component:
$$f_i(a) = h_i(a) \quad \forall a \in X\ \forall i$$
$\Rightarrow (f_1(a), \dots, f_m(a)) = (g_1(a), \dots, g_m(a))$ $\forall a \in X$ $\Rightarrow \phi = \psi$

Surjectivity: Let $F : \mathbb{k}[Y] \to \mathbb{k}[X]$ be a homomorphism of k-algebras. We have to show that there is a polynomial map $\phi : X \to Y$ s.t. $F = \phi^*$.

$X \subseteq \mathbb{A}^n$, $Y \subseteq \mathbb{A}^m$

Then $\mathbb{k}[Y] = \mathbb{k}[x_1, \dots, x_m]/I(Y)$.

For $i = 1, \dots, m$ denote $\overline{x_i} = x_i + I(Y) \in \mathbb{k}[Y]$. As a function on Y, $\overline{x_i}$ is the projection to the i-th component.

Define $g_i := F(\overline{x_i}) \in \mathbb{k}[X]$ for $i = 1, \dots, m$. Consider g_i as polynomial functions on X

$$g : X \longrightarrow \mathbb{k}$$
$$a \longmapsto g_i(a)$$

Define $\phi : X \longrightarrow \mathbb{A}^m$
$$a \longmapsto (g_1(a), \dots, g_m(a))$$

$\phi(X) \subseteq Y$

Let $h \in I(Y)$ and $a \in X$ arbitrary. We have to show that $h(\phi(a)) = 0$.

$$h(\phi(a)) = h(g_1(a),\dots,g_m(a)) = h(g_1,\dots,g_m)(a) = 0$$

$$h(g_1,\dots,g_m) = h(F(\bar{x}_1),\dots,F(\bar{x}_m))$$

F homomorphism of algebras, h is a polynomial

$$= F(h(\bar{x}_1,\dots,\bar{x}_m))$$

$$= F(\underbrace{\underbrace{h(x_1,\dots,x_m)}_{\in I(Y)} + I(Y)}_{=0 \text{ in } \mathbb{k}[x_1,\dots,x_m]/I(Y) = \mathbb{k}[Y]}) = 0$$

$F = \phi^* \iff F(f) = \phi^*(f) \quad \forall f \in \mathbb{k}[Y]$.

F and ϕ^* are algebra homomorphisms, therefore it is enough to check the equality $F(f) = \phi^*(f)$ on the generators o $\mathbb{k}[Y]$, i.e. on $\bar{x}_1, \bar{x}_2, \dots, \bar{x}_m$.

$F(\bar{x}_i) = \phi^*(\bar{x}_i)$

For each $a \in X$ we have $\phi^*(\bar{x}_i)(a) = \bar{x}_i(\phi(a))$

$\Rightarrow \bar{x}_i(g_1(a),\dots,g_m(a)) = g_i(a) = F(\bar{x}_i)(a)$.

projection to ith component

$\Rightarrow \phi^*(\bar{x}_i) = F(\bar{x}_i)$. ∎

Corollary: There is a contravariant functor

$\{\text{affine varieties}\} \longrightarrow \{\text{Finitely generated reduced } \mathbb{k}\text{-algebras}\}$

morphisms: polynomial maps

On objects: $X \longmapsto \mathbb{k}[X]$

On morphisms: $(\phi: X \to Y) \longrightarrow (\phi^*: \mathbb{k}[Y] \longrightarrow \mathbb{k}[X])$.

Moreover, the following version of surjectivity holds: If A is any finitely generated reduced $\mathbb{k}$-algebra, then there exists an affine variety X s.t. $\mathbb{k}[X] \cong A$.

Proof: We already proved all properties of the functor. Let A be a finitely generated reduced $\mathbb{k}$-algebra. Let A be generated by $a_1, \dots, a_n$. Then we have an algebra homomorphism $\Bbbk[x_1, \dots, x_n] \longrightarrow A$, $x_i \mapsto a_i$ $\forall i$.

$\Rightarrow A \cong \Bbbk[x_1, \dots, x_n]/I$ for some ideal I

A is reduced $\Rightarrow I$ is a radical ideal

By the Nullstellensatz: $I = I_a(X)$ for some affine variety $\underset{\substack{\| \\ V(I)}}{X} \subseteq \mathbb{A}^n$

$$\Rightarrow A \cong \Bbbk[x_1, \dots, x_n]/I(X) = \Bbbk[X] \qquad \square$$

Remark: The above functor induces a functor:

$$\left\{ \begin{matrix} \text{isomorphism classes} \\ \text{of affine varieties} \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} \text{isomorphism classes of} \\ \text{finitely generated reduced} \\ \Bbbk\text{-algebras} \end{matrix} \right\}$$

which is bijective on objects.

Another corollary of the theorem:

Corollary: Let $\phi: X \to Y$ be a polynomial map. Then ϕ is an isomorphism ⟺ $\phi^*: \Bbbk[Y] \to \Bbbk[X]$ is an isomorphism.
of affine varieties — of algebras

Proof: (⟹): Suppose ϕ is an isomorphism. Then $\exists\, \psi: Y \to X$ polynomial map s.t. $\phi \circ \psi = id_Y$ and $\psi \circ \phi = id_X$.

By one of the lemmas before the theorem:

$$\phi^* \circ \psi^* = (\psi \circ \phi)^* = id_X^* = id_{\Bbbk[X]}$$

The same argument shows $\psi^* \circ \phi^* = id_{\Bbbk[Y]} \Rightarrow \phi^*$ is an isomorphism of algebras.

(⟸): Assume that $\phi^*: \Bbbk[Y] \to \Bbbk[X]$ is an isomorphism. Then $\exists$ an algebra homomorphism $F: \Bbbk[X] \longrightarrow \Bbbk[Y]$ s.t.
$\phi^* \circ F = id_{\Bbbk[X]}$, $F \circ \phi^* = id_{\Bbbk[Y]}$.

By surjectivity in the theorem $\exists$ a polynomial map

$\psi : Y \longrightarrow X$ s.t. $F = \psi^*$.

$$(\phi\circ\psi)^* = \psi^*\circ\phi^* = F\circ\phi^* = id_{\Bbbk[Y]}$$

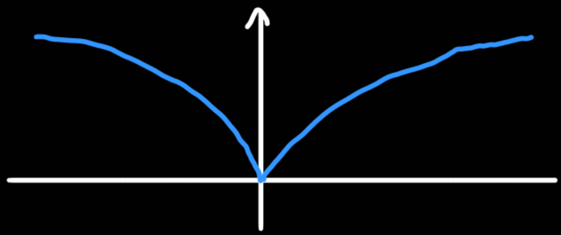
Also: $id_Y^* = id_{\Bbbk[Y]}$. By the injectivity of the theorem $\phi\circ\psi = id_Y$. The same argument gives $\psi\circ\phi = id_X$

$\Rightarrow \phi$ is an isomorphism of varieties. $\blacksquare$

7. november 2025

<u>Example</u>: $X = V(x^2-y^3) \subseteq \mathbb{A}^2$ $\quad \phi : \mathbb{A}^1 \longrightarrow X$

$$t \longmapsto (t^3, t^2)$$



We will show that ϕ is not an isomorphism by showing that ϕ^* is not an isomorphism.

$$\phi^* : \Bbbk[X] \longrightarrow \Bbbk[\mathbb{A}^1] = \Bbbk[t]$$

We will show that it is not surjective. The image of ϕ^* is generated by the images of generators of $\Bbbk[X]$.

$\Bbbk[X] = \Bbbk[x,y]/(x^2,y^3)$ is generated by $x + (x^2-y^3)$ and $y + (x^2-y^3)$.

What is $\phi^*(x + (x^2-y^3))$?

Geometrically, $x + (x^2-y^3)$ is a projection to the first component, so $\phi^*(x+(x^2-y^3)) = \phi^*(\pi_1) = \pi_1\circ\phi$.

For each $a\in\Bbbk$ we have $\pi_1(\phi(a)) = \pi_1(a^3,a^2) = a^3$

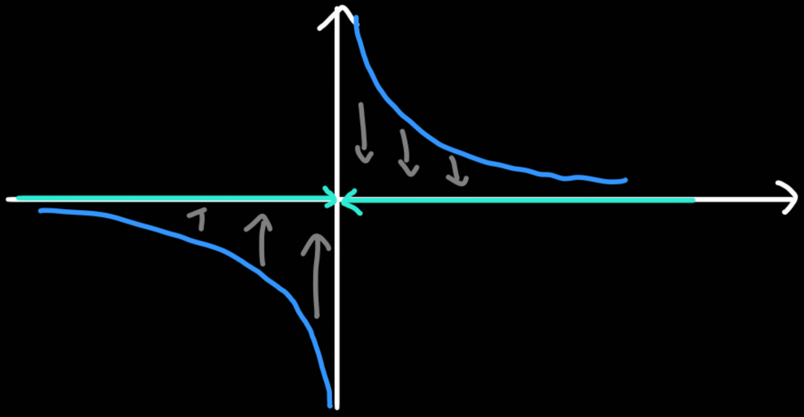
$\Rightarrow \phi^*(\pi_1) = \phi^*(x+(x^2-y^3)) = t^3$

Similarly $\phi^*(y+(x^2-y^3)) = t^2$

$\Rightarrow$ The image of ϕ^* is generated by t^3 and t^2, so it does not contain t. $\Rightarrow \phi^*$ is not surjective $\Rightarrow \phi^*$ is not an isomorphism

Definition: A map $\phi : X \longrightarrow Y$ is dominant if the image $\phi(X)$ is dense in Y.

We can have dominant polynomial maps that are not surjective. For example, projection $V(xy-1) \longrightarrow \mathbb{A}^1$



Proposition: Let $\phi : X \longrightarrow Y$ be a polynomial map. Then
(1) ϕ^* is injective $\Longleftrightarrow$ ϕ is dominant
(2) If ϕ^* is surjective, then ϕ is injective.

The previous example shows that the converse of (2) does not hold.

Proof: (1) ϕ^* is a ring homomomorphism, so ϕ^* is injective $\Longleftrightarrow \ker \phi^* = \underline{0}$.
$g \in \ker \phi^* \Longleftrightarrow \phi^*(g) = 0 \Longleftrightarrow g \circ \phi = 0 \Longleftrightarrow g(\phi(x)) = 0 \quad \forall x \in X$
$\Longleftrightarrow g|_{\phi(X)} = 0$

($\Leftarrow$): Suppose that ϕ is dominant. Then $\overline{\phi(X)} = Y$. So, if g is zero on $\phi(X)$, then it is zero also on $\overline{\phi(X)} = Y$, because g is continuous $\Rightarrow g = 0$, so $\ker \phi^* = 0$

($\Rightarrow$): Suppose ϕ is not dominant. Then $\overline{\phi(X)}$ is a proper subvariety of Y, so we can write it as $\overline{\phi(X)} = V_Y(f_1, \dots, f_r)$

For some $f_1, \dots, f_r \in \Bbbk[Y]$. Then f_1 vanishes on $\overline{\phi(X)}$, so $f_1|_{\phi(X)} = 0$
$\Rightarrow f_1 \in \ker \phi^* \Rightarrow \phi^*$ is not injective.

(2) Let $a = (a_1, \dots, a_n)$, $a' = (a_1', \dots, a_n') \in X$, $a \neq a'$. Then $\exists i.\ a_i \neq a_i'$. If π_i is the projection to the i-th component, then $\pi_i : X \longrightarrow \Bbbk$, so $\pi_i \in \Bbbk[X]$ and $\pi_i(a) \neq \pi_i(a')$. By assumption, ϕ^* is surjective, so there exists $g \in \Bbbk[Y]$ s.t. $\pi_i = \phi^*(g) = g \circ \phi$
$g(\phi(a)) = \pi_i(a) \neq \pi_i(a') = g(\phi(a')) \Rightarrow \phi(a) \neq \phi(a')$. ▨

<u>Definition</u>: Let $X \subseteq \mathbb{P}^n$ be a projective variety. The homogeneous coordinate ring of X is $S(X) = \Bbbk[x_0, x_1, \dots, x_n] / I_p(X)$.

Two properties of $S[X]$:

- $\Bbbk[x_0, x_1, \dots, x_n]$ is noetherian, so $S[X]$ is noetherian.
- $\Bbbk[x_0, x_1, \dots, x_n]$ is a graded ring and $I_p(X)$ is a homogeneous ideal, so $S[X]$ is a graded ring.

<u>Definition</u>: Let $X \subseteq \mathbb{P}^n$ be a projective variety.

(1) For a homogeneous ideal $J \trianglelefteq S[X]$ we define
$V_X(J) = \{x \in X \mid f(x) = 0 \text{ for each homogeneous } f \in J\}$.
This is a projective subvariety of X.

(2) For each subset $Y \subseteq X$ we define the ideal of Y in $S[X]$ by $I_X(Y) = (f \in S[X] \text{ homogeneous} \mid f(x) = 0\ \forall x \in Y)$.

As in the affine case, we have:

- If Y is a subvariety, then $V_X(I_X(Y)) = Y$
- If $J \trianglelefteq S[X]$ is homogeneous and the radical of J is not the irrelevant ideal of $S[X]$, then $I_X(V_X(J)) = \sqrt{J}$

[Projective Relative Nullstellesatz]

2. Regular Functions

Definition: Let $X \subseteq \mathbb{A}^n$ be an affine variety and $U \subseteq X$ an open subset. A regular function on U is a map $\phi: U \to \mathbb{k}$ such that for each $a \in U$ there exists an open neighbourhood U_a of a in U and there exist $p_a, q_a \in \mathbb{k}[x]$ such that $q_a(x) \neq 0$ for $x \in U_a$ and $\phi(x) = \frac{p_a(x)}{q_a(x)}$ for each $x \in U_a$.

The quotient $\frac{p_a(x)}{q_a(x)}$ is not necessarily globaly defined on U.

Example: $X = V(x_1x_4 - x_2x_3) \subseteq \mathbb{A}^4$. This is an irreducible hypersurface in $\mathbb{A}^4$. This is a set of all 2×2 singular matrices. Let $U = X \setminus V(x_2, x_4) = \{(a_1,a_2,a_3,a_4) \in \mathbb{A}^4 \mid x_1x_4 = x_2x_3,\ a_2 \neq 0 \text{ or } a_4 \neq 0\}$ be the set of all singular matrices with the second column nonzero.

$$\begin{aligned} \phi: U &\longrightarrow \mathbb{k} \\ (a_1,a_2,a_3,a_4) &\longmapsto \begin{cases} \frac{a_1}{a_2}; & a_2 \neq 0 \\ \frac{a_3}{a_4}; & a_4 \neq 0 \end{cases} \end{aligned}$$

This is a well defined map, because $\frac{a_1}{a_2} = \frac{a_3}{a_4}$ if $(a_1,a_2,a_3,a_4) \in X$ $a_2 \neq 0$ and $a_4 \neq 0$. It is a regular function, but neither $\frac{a_1}{a_2}$ nor $\frac{a_3}{a_4}$ is defined everywhere on U.

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Definition: Let $X \subseteq \mathbb{P}^n$ be a projective variety and $U \subseteq X$ an open subset. A regular function on U is a map $\phi: U \to \mathbb{k}$ satisfying the following property: For each $a \in U$ there exists an open neighbourhood U_a of a in U and there exist homogeneous polynomials $f_a, g_a \in \mathbb{k}[x_0, x_1, \dots, x_n]$ of the same degree such that $g_a(x) \neq 0$ for each $x \in U_a$ and $\phi(x) = \frac{f_a(x)}{g_a(x)}$.

Definition is well-defined: If f_a, g_a are of degree d, then for each $\lambda \neq 0$ we have $\frac{f_a(\lambda x_0, \dots, \lambda x_n)}{g_a(\lambda x_0, \dots, \lambda x_n)} = \frac{\lambda^d f_a(x_0, \dots, x_n)}{\lambda^d g_a(x_0, \dots, x_n)}$, so the map $x \mapsto \frac{f_a(x)}{g_a(x)}$ is well defined.

Each affine variety $X \subseteq \mathbb{A}^n$ is an open subset of a projective variety $\overline{X} \subseteq \mathbb{P}^n$. Last time we had a definition of a regular function on an (open subset of an) affine variety. Is this definition equivalent to the above? Yes.

Suppose $X \subseteq \mathbb{A}^n \cong U_0 \subseteq \mathbb{P}^n$ is an affine variety, and $U \subseteq X$ is an open subset. Assume that $\phi : U \longrightarrow \mathbb{k}$ is a regular map according to the definition from last time. For each $a \in U$ there exists an open neighbourhood U_a of a in $U \subseteq U_0 \cong \mathbb{A}^n$ and there exist polynomials $f_a, g_a \in \mathbb{k}[x_1, \dots, x_n]$ such that $\phi(x) = \frac{f_a(x)}{g_a(x)}$ for each $x \in U_a$. Let $d = \max\{\deg f_a, \deg g_a\}$. Define $F(x_0, \dots, x_n) = x_0^d f_a\left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right)$, $G(x_0, \dots, x_n) = x_0^d g_a\left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right)$. Then F and G are homogeneous polynomials of the same degree d and for each $x = (1 : x_1 : \dots : x_n)$ we have

$$\frac{F(1, x_1, \dots, x_n)}{G(1, x_1, \dots, x_n)} = \frac{f_a(x_1, \dots, x_n)}{g_a(x_1, \dots, x_n)} = \phi(x)$$

$\Rightarrow \phi$ is regular according to the new definition.
The converse can be shown the same way, using dehomogenization.

We will often use only the second definition for quasiprojective varieties (= open subsets of projective varieties), this includes open subsets of affine varieties.

Sometimes we will also need regular functions on open subsets of varieties in $\mathbb{P}^n \times \mathbb{P}^m$. Let X be a closed subset of $\mathbb{P}^m \times \mathbb{P}^n$ and U an open subset of X. A regular function on U is a map $\phi: U \longrightarrow \mathbb{k}$ such that for each $a \in V$ there exist an open neighbourhood V_a of a in V and polynomials $f_a, g_a \in \mathbb{k}[x_0, \dots, x_m, y_0, \dots, y_n]$ that are homogeneous of the same degree in $x_0, \dots, x_m$ and homogeneous of the same degree in $y_0, \dots, y_n$ s.t. $\forall x \in V_a.\ g_a(x) \neq 0$ and $\phi(x) = \frac{f_a(x)}{g_a(x)}$.

Lemma: Let X be a quasiprojective variety and U an open subset of X. Then the set $\mathcal{O}_X(U)$ of all regular functions on U is a $\mathbb{k}$-algebra for point-wise operations.

Proof: The only question is why the sum and the product of regular functions is a regular function. We show this for the sum. Let $\phi_1, \phi_2: U \longrightarrow \mathbb{k}$ be two regular functions. Let $a \in U$ be arbitrary. Then there exist open neighbourhoods V_1, V_2 of a in U and homogeneous polynomials of the same degree d_1, $f_1, g_1 \in \mathbb{k}[x_0, \dots, x_n]$ and homogeneous polynomials of the same degree d_2, $f_2, g_2 \in \mathbb{k}[x_0, \dots, x_n]$ s.t.

$$\phi(x) = \frac{f_1(x)}{\underset{\neq 0}{g_1(x)}} \ \forall x \in V_1, \qquad \phi(x) = \frac{f_2(x)}{\underset{\neq 0}{g_2(x)}} \ \forall x \in V_2.$$

Let $U_a = V_1 \cap V_2$. Then $g_1(x) g_2(x) \neq 0$ on U_a, and

$$\phi_1(x) + \phi_2(x) = \frac{f_1(x)}{g_1(x)} + \frac{f_2(x)}{g_2(x)} = \frac{f_1(x) g_2(x) + f_2(x) g_1(x)}{g_1(x) g_2(x)}.$$

and the numerator and denominator are homogeneous of the same degree $d_1 + d_2$.

Remark: In the definitions of regular maps, we allow that the numerator is 0.

$\Rightarrow \phi_1 + \phi_2$ is a regular map on U.
The same for $\phi_1 \phi_2$. ▨

Remark: $\mathcal{O}_X(U)$ is not necessarily finitely generated. The first counterexamples were constructed by Rees and Nagata. These counterexamples are among the few non-noetherian rings that we will consider, but we will never work with ideals of $\mathcal{O}_X(U)$.

Lemma: Let X be a quasiprojective variety and ϕ a regular function on X. Then the set $V(\phi) := \{x \in X \mid \phi(x) = 0\}$ is closed in X.

Proof: For each $a \in X$ there exists an open neighborhood V_a of a in X and homogeneous polynomials of the same degree f_a, g_a s.t. $g_a(x) \neq 0$ $\forall x \in V_a$ and $\phi(x) = \frac{f_a(x)}{g_a(x)}$ $\forall x \in X$.

$$V_a \setminus V(\phi) = V_a \setminus V(f_a) = \underbrace{V_a}_{\text{open in } X} \cap \underbrace{(X \setminus V(f_a))}_{\text{open in } X}$$

$\Rightarrow V_a \setminus V(\phi)$ is open in X for each $a \in X$

$\Rightarrow \bigcup_{a \in X} (V_a \setminus V(\phi)) = X \setminus V(\phi)$ is open in X. $\Rightarrow V(\phi)$ is closed in X. ∎

Corollary: Let X be an irreducible quasiprojective variety, U an open subset of X and $\phi, \psi : X \longrightarrow \mathbb{k}$ regular functions that agree on U. Then $\phi = \psi$ on X.

Proof: $\phi(x) = \psi(x)$ $\forall x \in U \Rightarrow U \subseteq V(\phi - \psi)$. By the lemma $V(\phi - \psi)$ is closed in X, so $\bar{U} \subseteq V(\phi - \psi)$. X is irreducible, so $\bar{U} = X$ and $X \subseteq V(\phi - \psi)$, which means $\phi(x) = \psi(x)$ $\forall x \in X$. ∎

Let X be a quasiprojective variety and $U \subseteq V$ be an open subset of X. For each $\phi \in \mathcal{O}_X(V)$ the restriction $\phi|_U$ is a regular function on U, so $\phi|_U \in \mathcal{O}_X(U)$. So we have a restriction map $res_{V|U} : \mathcal{O}_X(V) \longrightarrow \mathcal{O}_X(U)$

$$\phi \longmapsto \phi|_V$$

This map satisfies the following two properties:

- $res_{V|V} = id_{\mathcal{O}_X(V)}$
- If $U \subseteq V \in W$ are open subsets of X, then $res_{U|V} \circ res_{W|V} = res_{W|V}$.

<u>Definition</u>: Let X be a topological space. A presheaf (slovene: predsnop) $\mathcal{F}$ on X consists of the following data:

(1) For each open subset $U \subseteq X$ we have a set $\mathcal{F}(U)$. Its elements are called sections on U. If $U = X$, they are called global sections.

(2) For each pair of open subsets $U, V \subseteq X$ satisfying $U \subseteq V$ we have a map $res_{V|U} : \mathcal{F}(V) \longrightarrow \mathcal{F}(U)$, called restriction, which satisfies the following two properties:

(a) $res_{U|U} = id_{\mathcal{F}(U)}$ for each open set U in X.

(b) If $U \subseteq V \subseteq W$ are open subsets of X, then

$$res_{W|U} = res_{V|U} \circ res_{W|U}.$$

We showed that $\mathcal{O}_X$ is a presheaf on X: For each open U we defined the set $\mathcal{O}_X(U)$ of regular functions on U, res is the usual restriction and (a), (b) are satisfied.

<u>Definition</u>: Let X be a topological space. A sheaf (slovene: snop) is a presheaf $\mathcal{F}$ on X satisfying the gluing property (lastnost lepljenja): For each open subset $U \subseteq X$ and each open cover $\{U_i \mid i \in I\}$ of U and each collection of sections $\phi_i \in \mathcal{F}(U_i)$ $(i \in I)$ satisfying

$$res_{U_i|U_i \cap U_j}(\phi_i) = res_{U_j|U_i \cap U_j} \qquad \forall i, j$$

there exists a unique $\phi \in \mathcal{F}(U)$ s.t. $res_{U|U_i}(\phi) = \phi_i$.

Let's check that $\mathcal{O}_X$ is a sheaf. Let $U \subseteq X$ be an open subset, $\{U_i \mid i \in I\}$ open cover of U and for each i let ϕ_i be a regular function on U_i ($\phi_i \in \mathcal{O}_X(U_i)$), such that $\phi_i|_{U_i \cap U_j} = \phi_j|_{U_i \cap U_j}$ $\forall i,j$. Because of the last equality we have a well defined function $\phi : U \longrightarrow \mathbb{k}$, $x \in U_i$, $x \longmapsto \phi_i(x)$. We have to show that ϕ is a regular function. Let $a \in U$ be an arbitrary point. Then $a \in U_i$ for some i. ϕ_i is a regular function on U_i, so there exists an open neighborhood $V_a \subseteq U_i$ and homogeneous polynomials of the same degree f_a, g_a s.t. $g_a(x) \neq 0$ for $x \in V_a$ and $\phi_i(x) = \frac{f_a(x)}{g_a(x)}$ $\forall x \in V_a$. Then V_a is also open in U and $\phi(x) = \frac{f_a(x)}{g_a(x)}$ $\forall x \in V_a$.
$\Rightarrow \phi$ is a regular function on U. It is also clear $\phi|_{U_i} = \phi_i$ $\forall i$.
$\Rightarrow \mathcal{O}_X$ is a sheaf on X. We call it the structure sheaf of X.
strukturni snop

We can define sheaves and presheaves in a categorical way. Let X be a topological space. Consider the category $\mathcal{C}$ of all open subsets of X. If $U \subseteq V$ are open subsets of X, then there is a unique morphism from U to V: $U \hookrightarrow V$. If $U \nsubseteq V$, then the set of morphisms from U to V is empty. The compositions are defined in the obvious way.
Then a presheaf on X is a contravariant functor from $\mathcal{C}$ to Set, and a sheaf is a contravariant functor satisfying an additional (gluing) property.
We could also look at contravariant functors from $\mathcal{C}$ to categories of groups, rings, modules, ... We get (pre)sheaves of groups, rings, modules, ... This means that $\mathcal{F}(U)$ is a group/ring/module ... for each open subset U of X and that the restriction maps are homomorphisms of groups/rings/modules ...
We showed that $\mathcal{O}_X(U)$ is a ring for each open subset

U of a quasiprojective variety X. The restrictions are ring homomorphisms $\Rightarrow$ $\mathcal{O}_X$ is a sheaf of rings on the quasi-projective variety X.

This will be important later, when we will consider schemes.

Recall that a distinguished open subset of an affine variety $X \subseteq \mathbb{A}^n$ is a set of the form $D(f) := \{x \in X \mid f(x) \neq 0\}$ where $f \in \mathbb{k}[X]$

Theorem: Let X be an affine variety and $f \in \mathbb{k}[X]$. Then:
(1) $\mathcal{O}_X(X) = \mathbb{k}[X]$.
(2) $\mathcal{O}_X(D(f)) = \{ \frac{g}{f^m} \mid g \in \mathbb{k}[X],\ m \in \mathbb{N}_0 \}$.
In particular, regular functions on a distinguished open set an affine variety are everywhere defined quotients of two polynomial functions.

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Proof: (1) is a special case of (2) if we take $f=1$. We have to prove only (2).
($\supseteq$): If $x \in D(f)$, then $f(x)^m \neq 0$, so $\frac{g(x)}{f(x)^m}$ is well defined, and $x \mapsto \frac{g(x)}{f^m(x)}$ is clearly a regular function on $D(f)$. This proves the second part of (2): $x \mapsto \frac{g(x)}{f(x)^m}$ is everywhere defined on $D(f)$.
($\subseteq$): Let $\phi: D(f) \longrightarrow \mathbb{k}$ be a regular function. For each $a \in D(f)$ there exists an open neighborhood V_a of a in $D(f)$ and there exist polynomial functions $p_a, q_a \in \mathbb{k}[X]$ such that for each $x \in V_a$ we have $q_a(x) \neq 0$ and $\phi(x) = \frac{p_a(x)}{q_a(x)}$.
We first make some reductions so that we will get nicer V_a, p_a, q_a.
The assumptions do not change if we take a smaller

neighbourhood. Each open set is a union of distinguished open sets, therefore we may assume that U_a is a distinguished open set, so of the form $U_a = D(r_a)$ for some $r_a \in \mathbb{k}[X]$.

On $D(r_a)$ we have $r_a(x) \neq 0$, so $\phi(x) = \frac{p_a(x)}{q_a(x)} = \frac{p_a(x) r_a(x)}{q_a(x) r_a(x)}$ for all $x \in D(r_a)$. We can replace p_a by $p(a)r(a)$ and $q(a)$ by $q(a)r(a)$ and the assumptions still hold, so we may assume that $p_a(x) = q_a(x) = 0$ for $x \in V_X(r_a) = X \setminus D(r_a)$.

$\Rightarrow D(r_a) = D(q_a)$, $V_X(r_a) = V_X(q_a)$.

Let $a, b \in D_f$ be different points. We decompose $D(f)$ as a union $D(f) = (D(r_a) \cap D(r_b)) \cup (D(r_a) \cap V_X(r_b)) \cup (V_X(r_a) \cap D(r_b)) \cup$

$$\cup\, (D(f) \cap V_X(r_a) \cap V_X(r_b))$$

If $x \in D(r_a) \cap D(r_b)$, then

$$\phi(x) = \frac{p_a(x)}{q_a(x)} = \frac{p_b(x)}{q_b(x)} \Rightarrow p_a(x) q_b(x) = p_b(x) q_a(x).$$

The equality also holds on $V_X(r_a)$ and on $V_X(r_b)$, since by the last reduction we have $p_a(x) = q_a(x) = 0$ on $V_X(r_a)$ and $p_b(x) = q_b(x) = 0$ on $V_X(r_b)$. So we have $p_a(x) q_b(x) = p_b(x) q_a(x)$ for each $x \in D(f)$. For each $a \in D(f)$, $U_a = D(r_a)$ is a neighbourhood of a in $D(f)$, so $D(f) = \bigcup_{a \in D(f)} D(r_a)$ $\quad /^c$

$$V_X(f) = \bigcap_{a \in D(f)} V_X(r_a) = \bigcap_{a \in D(f)} V_X(q_a) = V_X\Big(\bigcup_{a \in D(f)} \{q_a\}\Big) = V_X(\{q_a \mid a \in D(f)\}).$$

We apply $I_X(\cdot)$ to this equality:

$$I_X(V_X(f)) = I_X(V_X(\{q_a \mid a \in D(f)\})) = \sqrt{(q_a \mid a \in D(f))}$$

relative Nullstellensatz

$f \in I_X(V_X(f)) \Rightarrow \exists m \in \mathbb{N}.\ f^m \in (q_a \mid a \in D(f))$.

By the definition of an ideal generated by some set there exist finitely many elements q_a and $h_a \in \mathbb{k}[X]$ such that $f^m = \sum_{\substack{\text{finite} \\ a \in A}} h_a q_a$. We define $g = \sum_{a \in A} h_a p_a$.

So we have $p_a(x) g_b(x) = p_b(x) g_a(x)$ for each $x \in D(f)$. We have to show that $\phi(x) = \frac{g(x)}{f(x)^m}$ for each $x \in D(f)$. Let $b \in D(f)$ be arbitrary. For all $x \in U_b = D(r_b)$ we have $\phi(x) = \frac{p_b(x)}{g_b(x)}$.

$$g(x)\, g_b(x) = \sum_{a \in A} h_a(x) p_a(x) g_b(x) = \sum_{a \in A} h_a(x) g_a(x) p_b(x) = f(x)^m p_b(x)$$

On $D(r_b)$ we have $g_b(x) \neq 0$ and $f(x) \neq 0$, so $\frac{g(x)}{f(x)^m} = \frac{p_b(x)}{g_b(x)} = \phi(x)$ for all $x \in U_b$. Since $b \in D(f)$ was arbitrary, we get $\phi(x) = \frac{g(x)}{f(x)^m}$ for all $x \in D(f)$. ▨

<u>Definition</u>: Let R a (commutative) ring. Let S be a set that is multiplicatively closed ($a \in S$, $b \in S \Rightarrow ab \in S$) and contains 1. On $R \times S$ we define a relation

$$(a,b) \sim (b,t) :\Longleftrightarrow \exists u \in S.\ u(at - bs) = 0$$

This is an equivalence relation. We denote the equivalence class $[(a,s)]$ with $\frac{a}{s}$, and we denote the quotient set $(R \times S)/_\sim$ by $S^{-1}R$.
We define adition and multiplication on $S^{-1}R$ by

$$\frac{a}{s} + \frac{b}{t} = \frac{at + bs}{st}, \qquad \frac{a}{s} \cdot \frac{b}{t} = \frac{ab}{st}$$

We can check that these operations are well defined and that $S^{-1}R$ is a ring for these operations.
We call it the ring of fractions of R.

<u>Remark</u>: If R is noetherian, then $S^{-1}R$ is noetherian.

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<u>Examples</u>: 1) If R is a domain (without zero-divisors) and $S = R \setminus \{0\}$, then $S^{-1}R$ is the field of fractions of R.
2) Let $a \in R$ be an element that is not nilpotent and $S = \{1, a, a^2, \dots\}$. Then S is multiplicatively closed and we can define $S^{-1}R$.

$$S^{-1}R := \left\{ \frac{x}{a^n} \mid x \in R,\ n \in \mathbb{N}_0 \right\}.$$

This ring is usually denoted by R_a or $R[\frac{1}{a}]$. We will use the notation $R[\frac{1}{a}]$.
Concrete example: $R=\mathbb{Z}$, $a=2 \Rightarrow S^{-1}R$ are all fractions where the denominator is a power of 2.
Remark: We assume that a is not nilpotent, as otherwise $S^{-1}R$ is a trivial ring.
3) If $p \trianglelefteq R$ is a prime ideal, then $S=R\setminus p$ is a multiplicatively closed set with 1. The ring of fractions $S^{-1}R$ is in this case denoted by R_p and called the localization of R at p.
Prime ideals in R_p are of the form $S^{-1}Q = \{\frac{a}{s} \mid a\in Q, s\in S=R\setminus p\}$ where Q is a prime ideal of R and contained in P.
$\Rightarrow S^{-1}P$ is a maximal ideal.
The set of all elements from S are invertible in $S^{-1}R$
$\Rightarrow S^{-1}P = P_p$ is the unique maximal ideal of R_p.

<u>Definition</u> A ring is local if it has a unique maximal ideal.

<u>Example</u>: Localizations of R at prime ideals of R are local rings.

<u>Corollary</u>: Let X be an affine variety and $f\in \mathbb{k}[X]$. Then $\mathcal{O}_X(D(f)) \cong \mathbb{k}[X][\frac{1}{f}]$.

<u>Proof</u>: We define a map

$$\Phi: \mathbb{k}[X][\tfrac{1}{f}] \longrightarrow \mathcal{O}_X(D(f))$$
$$\frac{g}{f^m} \longmapsto \left(x \longmapsto \frac{g(x)}{f(x)^m}\right)$$

If $x\in D(f)$, then $f(x)^m \neq 0$, so $\frac{g(x)}{f(x)^m}$ is defined, so $x\mapsto \frac{g(x)}{f(x)^m}$ is a regular function on $D(f)$.
We have to show that Φ is well defined. Suppose we have $\frac{g}{f^m} = \frac{h}{f^n}$ in $\mathbb{k}[X][\frac{1}{f}]$. $S=\{1, f, f^2, \dots\}$
By definition then there exists $k\in\mathbb{N}_0$ such that $(gf^n - hf^m)f^k = 0$

in $\mathbb{k}[X] \Rightarrow (g(x)f(x)^n - h(x)f^m(x))\cdot f(x)^k = 0 \quad \forall x \in X.$

If $x \in D(f)$, then $f(x) \neq 0$ and we get

$$g(x)f(x)^n = h(x)f^m(x) \Rightarrow \frac{g(x)}{f(x)^m} = \frac{f(x)}{f(x)^n} \quad \forall x \in D(f)$$

$\Rightarrow \Phi$ is well defined.

Clearly Φ is a homomorphism of $\mathbb{k}$-algebras. It is surjective by the theorem from last time.

Injectivity of Φ:

Assume that $\frac{g}{f^m} \in \ker\Phi$. This means $\frac{g(x)}{f(x)^m} = 0 \ \forall x \in D(f) \Rightarrow$

$\Rightarrow g(x) = 0 \ \forall x \in D(f) \Rightarrow g(x)\cdot f(x) = 0 \ \forall x \in X \Rightarrow gf = 0$ in $\mathbb{k}[X]$.

$$(g\cdot 1 - 0\cdot f^m)\cdot f = 0 \quad \text{in } \mathbb{k}[X].$$

By the definition of fractions we get $\frac{g}{f^m} = \frac{0}{1}$ in $\mathbb{k}[X][\frac{1}{f}]$.

$\Rightarrow$ The kernel is trivial $\Rightarrow \Phi$ is injective. $\blacksquare$

3. Regular maps

Definition 1: Let X be a quasiprojective variety, $Y \subseteq \mathbb{A}^n$ an affine variety and $U \subseteq Y$ an open subset. The maps $\phi: X \to U$ is a regular map if there exist regular functions $\phi_1, \dots, \phi_n$ on X such that $\phi(x) = (\phi_1(x), \dots, \phi_n(x))$ for each $x \in X$.

Lemma: Let X be a quasiprojective variety, $Y \subseteq \mathbb{A}^n$ an affine variety, $U \subseteq Y$ an open subset and $\phi: X \to U$ a regular map. Then ϕ is continuous in the Zariski topology.

The proof is the same as in the case of a polynomial, the only difference is that we use the fact that $V(\psi) = \{x \in X \mid \psi(x) = 0\}$ is closed in X if ψ is a regular function on X.

In the case when $Y \subseteq \mathbb{P}^n$ is a quasiprojective variety we cannot define a regular map $X \to Y$ as an $(n+1)$-tuple of regular functions, because we have to be carefull about common zeros.

Definition2: Let X, Y be quasiprojective varieties, $Y \subseteq \mathbb{P}^n$. A map $\phi: X \longrightarrow Y$ is called a regular map if for each $a \in X$ the following holds: For some index $i \in \{0, 1, \dots, n\}$ satisfying $\phi(a) \in U_i = \mathbb{P}^n \setminus V_p(x_i)$ there exists an open neighbourhood V of a in X such that $\phi(V) \subseteq U_i$ and the restriction $\phi|_V : V \longrightarrow U_i$ is regular according to the previous definition.

Remark 1: The definition is independent of the chosen index i: Let $a \in X$ be such that $\phi(a) \in U_i \cap U_j$. We use the definition for i: there exists an open neighbourhood U of a in X such that $\phi(U) \subseteq U_i$ and $\phi|_U : U \longrightarrow U_i \cong \mathbb{A}_n$ is according to Def. 1. By the previous lemma this restriction is continuous, so $U' = (\phi|_U)^{-1}(U_i \cap U_j)$ is open in U. $\phi(U') = (\phi|_U)(U') \subseteq U_j$. We want to show that $\phi|_{U'} : U' \longrightarrow U_j \cong \mathbb{A}^n$ is a regular map according to Definition 1. $\phi|_U$ is regular according to Definition 1, so there exist regular functions $\phi_0, \dots, \phi_n$ on U such that ϕ_i constantly equal to 1 and $\phi(x) = (\phi_0(x) : \dots : \phi_n(x))$ for each $x \in U$. $\phi(U') \subseteq U_j$, so $\phi_j(x) \neq 0$ $\forall x \in U'$, and for each $x \in U'$ we have

$$\phi|_{U'}(x) = \left(\frac{\phi_0(x)}{\phi_j(x)} : \dots : \frac{\phi_n(x)}{\phi_j(x)}\right),$$

$\frac{\phi_j(x)}{\phi_j(x)} = 1$, so $\phi|_{U'} : U' \longrightarrow U_j \cong \mathbb{A}^n$ is regular according to Def. 1.
$\Rightarrow$ Definition 2 is independent of i.

Remark 2: If Y is an (open subset of an) affine variety, then Definition 2 is equivalent to Definition 1.

Lemma: Regular maps are continuous in the Zariski topology.

Proof: Let X and $Y \subseteq \mathbb{P}^n$ be quasiprojective varieties and $\phi: X \to Y$ a regular map. Let Z be a closed subset of Y. We have to show that $\phi^{-1}(Z)$ is closed in X.
By the definition of a regular map for each $x \in X$ there exists an index i and an open neighbourhood U_x of x in X such that $\phi(U_x) \subseteq U_i$ and $\phi|_{U_x}: U_x \longrightarrow U_i \cong \mathbb{A}^n$ is regular according to Definition 1. For this restriction we can use the result that a regular map to an affine variety is continuous, so $(\phi|_{U_x})^{-1}(U_i \cap Z) = U_x \cap \phi^{-1}(Z)$ is closed in U_x.
$\Rightarrow U_x \setminus \phi^{-1}(Z)$ is open in U_x, so also open in X.
$\Rightarrow \bigcup_{x \in X} (U_x \setminus \phi^{-1}(Z)) = X \setminus \phi^{-1}(Z)$ is open in X
$\Rightarrow \phi^{-1}(Z)$ is closed in X. ▨

Corollary: Let X, Y be quasiprojective varieties, $Y \subseteq \mathbb{P}^n$. Then a map $\phi: X \to Y$ is regular $\Leftrightarrow$ it is continuous and the restriction $\phi^{-1}(U_i) \longrightarrow U_i \cong \mathbb{A}^n$ is regular by Def. 1 for each $i = 0, 1, \dots, n$.

Proposition: Let $X \subseteq \mathbb{P}^m$ and $Y \subseteq \mathbb{P}^n$ be quasiprojective varieties. A map $\phi: X \longrightarrow Y$ is regular $\Longleftrightarrow$ for each $a \in X$ there exists an open neighbourhood U_a of a in X and polynomials $f_0, \dots, f_n \in \mathbb{k}[x_0, \dots, x_n]$ that are homogeneous of the same degree such that for each $x \in U_a$ we have $f_i(x) \neq 0$ for at least one i and $\phi(x) = (f_0(x) : \dots : f_n(x))$.

Proof: ($\Rightarrow$): Let ϕ be a regular map and $a \in X$ arbitrary. Then there exists i s.t. $\phi(a) \in U_i$. By the definition of a regular map there exists an open neighbourhood U_a of a in X and regular functions $\phi_0, \dots, \phi_n$ on U_a with ϕ_i constantly 1 such that $\phi(x) = (\phi_0(x) : \dots : \phi_{i-1}(x) : 1 : \phi_{i+1}(x) : \dots : \phi_n(x))$ for each $x \in U_a$.

By the definition of regular functions there exists an open neighbourhood U_a' of a in U_a and there exist homogeneous polynomials $g_0, \dots, g_n, h_0, \dots, h_n \in \mathbb{k}[x_0, \dots, x_m]$, g_j and h_j of the same degree $\forall j$ such that $\phi_j = \frac{g_j(x)}{h_j(x)}$ $\forall x \in U_a$.

$$\Rightarrow \phi(x) = \left(\frac{g_0(x)}{h_0(x)} : \dots : \frac{g_{i-1}(x)}{h_{i-1}(x)} : 1 : \frac{g_{i+1}(x)}{h_{i+1}(x)} : \dots : \frac{g_n(x)}{h_n(x)}\right) \quad \forall x \in U_a'.$$

We clear the denominators and get polynomials $f_0, \dots, f_n \in \mathbb{k}[x_0, \dots, x_n]$ such that $\phi(x) = (f_0(x) : \dots : f_n(x))$ $\forall x \in U_a'$.

One can also check that $f_0, \dots, f_n$ don't have common zeroes on U_a'.

$(\Leftarrow)$: Similarly. $\blacksquare$

Corollary: Let $X \subseteq \mathbb{P}^n$ be a projective variety, $f_0, \dots, f_n \in \mathbb{k}[x_0, \dots, x_n]$, $U = X \setminus V_p(f_0, \dots, f_n)$ and $\phi(x) = (f_0(x) : \dots : f_n(x))$ for $x \in U$. Then $\phi: U \to \mathbb{P}^n$ is a regular map.

In general we cannot assume that polynomials $f_0, \dots, f_n$ from the proposition are defined globally on X, even if X is a projective variety.

Example: $X = \{(x:y:z) \in \mathbb{P}^2 \mid x^2 + y^2 = z^2\} \subseteq \mathbb{P}^2$

$\phi: X \longrightarrow \mathbb{P}^1$

$$(x:y:z) \longmapsto \begin{cases} (x : y-z) & \text{if } (x:y:z) \neq (0:1:1) \\ (y+z : x) & \text{if } (x:y:z) \neq (0:1:-1) \end{cases}$$

The definitions agree on the intersection, so ϕ is a regular map. Neither of the expressions is defined on the entire X.

21. november 2025

1 $X \subseteq \mathbb{P}^m$ kvaziprojektivna raznoterost, $\phi: X \longrightarrow \mathbb{P}^r \times \mathbb{P}^s$. ϕ je regularna preslikava, če za vsak $a \in X$ obstaja odprta okolica U_a od a v X in obstajajo homogeni polinomi iste stopnje $f_0, \dots, f_r \in \mathbb{k}[x_0, \dots, x_m]$ in obstajajo homogeni polinomi

iste stopnje $g_1, \dots, g_s \in \Bbbk[x_0, \dots, x_n]$, da za vsak $x \in U_a$ velja $\Phi(x) = ((f_0(x) : \dots : f_r(x)), (g_0(x) : \dots : g_s(x)))$ in $f_i(x) \neq 0$ za vsaj en i in $g_j(x) \neq 0$ za vsaj en j.

2) Naj bo X odprta podmnožica zaprte podmnožice v $\mathbb{P}^n \times \mathbb{P}^m$ in $\Phi : X \longrightarrow \mathbb{A}^r$. Φ je regularna preslikava, če obstajajo regularne funkcije $\phi_1, \dots, \phi_r$ na X, da za vsak $x \in X$ velja $\Phi(x) = (\phi_1(x), \dots, \phi_r(x))$.

3) Naj bo X odprta podmnožica zaprte podmnožice v $\mathbb{P}^n \times \mathbb{P}^m$ in $\Phi : X \longrightarrow \mathbb{P}^r$. Φ je regularna preslikava, če za vsak $a \in X$ obstaja okolica U_a od a v X in obstajajo polinomi $f_0, \dots, f_r \in \Bbbk[x_0, \dots, x_n, y_0, \dots, y_m]$, ki so homogeni iste stopnje v $x_0, \dots, x_n$ in homogeni iste stopnje v $y_0, \dots, y_m$, da za vsak $(x,y) \in U_a$ velja $\Phi(x,y) = (f_0(x,y) : \dots : f_r(x,y))$ in $f_i(x,y) \neq 0$ za vsaj en i.

4) Naj bo X odprta podmnožica v $\mathbb{P}^n \times \mathbb{P}^m$ in $\Phi : X \to \mathbb{P}^r \times \mathbb{P}^m$. Φ je regularna preslikava, če velja kot v 1), pri čemer so $f_1, \dots, f_r \in \Bbbk[x_0, \dots, x_n, y_0, \dots, y_n]$ homogeni iste stopnje v $x_0, \dots, x_n$ in homogeni iste stopnje v $y_0, y_1, \dots, y_m$, in enako velja za $g_0, \dots, g_s$.

Vse te preslikave so zvezne v topologiji Zariskega.
Na vajah: $\mathbb{P}^r \times \mathbb{P}^s$ je izomorfna projektivni raznoterosti. Za dokaz tega potrebujemo zgornje definicije regularnih preslikav. Te definicije bomo potrebovali tudi, ko bomo obravnavali preslikave, povezane s projekcijami $\mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^n$.

Definicija: Naj bosta X in Y kvaziprojektivni raznoterosti. Pravimo, da sta X in Y izomorfni, če obstajata regularni preslikavi $\varphi: X \longrightarrow Y$ in $\psi: Y \longrightarrow X$, da je $\varphi \circ \psi = id_Y$ in $\psi \circ \varphi = id_X$. V tem primeru pravimo, da sta φ in ψ izomorfizma.

Enako kot pri polinomskih preslikavah lahko definiramo povlek regularne preslikave:

Lema: (1) Naj bo $\phi: X \longrightarrow Y$ regularna preslikava med kvaziprojektivnima raznoterostima. Potem obstaja preslikava $\phi^*: \mathcal{O}_Y(Y) \longrightarrow \mathcal{O}_X(X)$, $g \longmapsto g \circ \phi$, ki je homomorfizem algeber.
(2) Če je $\psi: Y \longrightarrow Z$ še ena regularna preslikava, potem je kompozitum $\psi \circ \varphi: X \longrightarrow Z$ tudi regularna preslikava in velja $(\psi \circ \varphi)^* = \varphi^* \circ \psi^*$.

Lema: Če je $\phi: X \longrightarrow Y$ izomorfizem kvaziprojektivnih raznoterosti, potem je $\phi^*: \mathcal{O}_Y(Y) \longrightarrow \mathcal{O}_X(X)$ izomorfizem algeber.

Opomba: Če sta X in Y izomorfni projektivni raznoterosti, potem njuna homogena koordinatna kolobarja nista nujno izomorfna (vaje).

Razširimo pojma afinih in projektivnih raznoterosti:

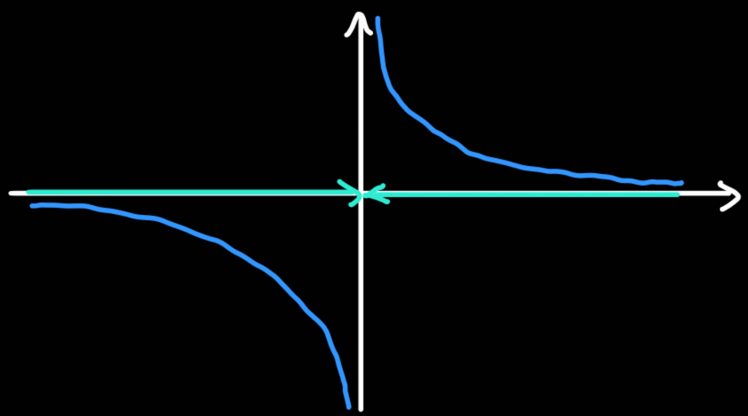
Definicija: (1) Kvaziprojektivna raznoterost, ki je izomorfna afini raznoterosti, rečemo afina raznoterost.

(2) Kvaziprojektivni raznoterosti, ki je izomorfna projektivni raznoterosti, rečemo projektivna raznoterost.

<u>Izrek</u>: Vsaka odlikovana odprta podmnožica afine raznoterosti je afina raznoterost.
Podoben dokaz na vajah. Dokaz na spletni učilnici.

<u>Posledica</u>: Naj bo X kvaziprojektivna raznoterost in $x \in X$. Potem ima x afino odprto okolico v X, t.j. okolico izomorfno afini raznoterosti.

<u>Primer</u>: $\mathbb{A}^1 \setminus \{0\}$ je izomorfen $V(xy-1) =: X$.



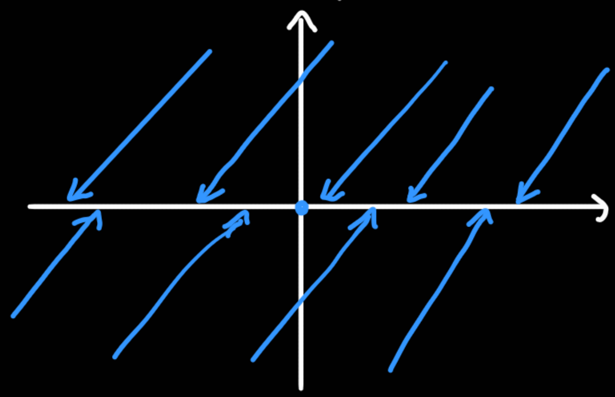
$$\left.\begin{array}{l} \pi: X \longrightarrow \mathbb{A}^1 \setminus \{0\} \text{ projekcija} \\ \varphi: \mathbb{A}^1 \setminus \{0\} \longrightarrow X \\ \qquad x \longmapsto (x, \frac{1}{x}) \end{array}\right\} \begin{array}{l} \text{regularni preslikavi,} \\ \text{oba kompozituma} \\ \text{sta identiteti} \end{array}$$

Ideja v splošnem: $D(f) \subseteq Z \subseteq \mathbb{A}^n \longrightarrow \mathbb{A}^{n+1}$

$$(a_1, \dots, a_n) \longmapsto \left(a_1, \dots, a_n, \frac{1}{f(a_1, \dots, a_n)}\right)$$

Kaj je slika regularne preslikave? Kakšne vrste množica je? Vemo, da v splošnem slika ni odprta in ni zaprta.

Imeli smo primer
$$\begin{aligned}\Phi: \mathbb{A}^2 &\longrightarrow \mathbb{A}^2\\ (x,y) &\longrightarrow (xy, x)\end{aligned}$$
Slika je $\mathbb{A}^1 \times (\mathbb{A}\setminus\{0\}) \cup \{(0,0)\}$.



<u>Definicija:</u> Množica X v $\mathbb{A}^n$ ali v $\mathbb{P}^n$ je **konstruktibilna**, če jo lahko zapišemo kot končno unijo $X = \bigcup_{i=1}^{m} (V_i \cap Z_i)$, kjer je V_i odprta, Z_i pa zaprta v $\mathbb{A}^n$ oziroma $\mathbb{P}^n$.

<u>Lema:</u> (1) Razred konstruktibilnih množic je najmanjši razred, ki vsebuje vse odprte množice in je zaprt za končne preseke in komplemente.
(2) Množica X je konstruktibilna $\Leftrightarrow$
$$X = Z_1 \setminus (Z_2 \setminus (Z_3 \setminus \cdots \setminus (Z_{m-1} \setminus Z_m) \cdots)),$$
kjer je $Z_1 \supseteq Z_2 \supseteq \cdots \supseteq Z_m$ padajoče zaporedje zaprtih množic v $\mathbb{A}^n$ oziroma $\mathbb{P}^n$.

25. november 2025

<u>Izrek [Chevalley]:</u> Naj bo $\Phi: X \longrightarrow Y$ regularna preslikava med kvaziprojektivnima raznoterostima. Tedaj Φ slika konstruktibilne množice v konstruktibilne množice.

<u>Posledica:</u> Če je $\Phi: X \longrightarrow Y$ regularna preslikava, potem je $\Phi(X)$ konstruktibilna množica.

<u>Izrek:</u> Projekcija $\pi: \mathbb{P}^n \times \mathbb{P}^m \longrightarrow \mathbb{P}^m$ je zaprta preslikava.

Dokaz: Naj bo $Z \subseteq \mathbb{P}^n \times \mathbb{P}^m$ zaprta množica. Radi bi dokazali, da je $\Phi(Z)$ zaprta. Z je zaprta v $\mathbb{P}^n \times \mathbb{P}^m$, zato je oblike $Z = V(f_1, \dots, f_r)$, kjer so polinomi $f_1, \dots, f_r \in \mathbb{k}[x_0, \dots, x_n, y_0, \dots, y_m]$ homogeni v $x_0, \dots, x_n$ in homogeni v $y_0, y_1, \dots, y_m$. Različna polinoma f_i in f_j sta lahko različne stopnje. Za vsak i naj bo f_i homogen stopnje d_i v $x_0, \dots, x_n$ in homogen stopnje e_i v $y_0, \dots, y_m$. Naj bo $d = \max\{e_i, d_i \mid i = 1, \dots, r\}$. Potem je

$$Z = V\left(x_i^{d-d_\ell} y_j^{d-e_\ell} f_\ell \mid i = 0, \dots, n, \, j = 0, \dots, m, \, \ell = 1, \dots, r\right\}$$

Zato lahko predpostavimo, da so $f_1, \dots, f_r$ homogeni iste stopnje d v $x_0, \dots, x_n$ in homogeni iste stopnje d v $y_0, \dots, y_m$.

Naj bo $a \in \mathbb{P}^m$ poljubna točka. Poiskali bomo zaprt pogoj za $a \in \pi(Z)$: a bo element $\pi(Z) \Leftrightarrow a$ bo element neke raznoterosti

Fiksirajmo homogene koordinate točke a: $a = (a_0 : \dots : a_n)$. Za vsak i definirajmo polinom $g_i(x_0, \dots, x_n) = f_i(x_0, \dots, x_n, a_0, \dots, a_m) \in \mathbb{k}[x_0, \dots, x_n]$. f_i je homogen stopnje d v spremenljivkah $x_0, \dots, x_n$, zato je g_i homogen polinom stopnje d.

Če bi za točko a vzeli homogene koordinate $(\lambda a_0, \dots, \lambda a_m)$, bi dobili polinom $f_i(x_0, \dots, x_n, \lambda a_0, \dots, \lambda a_m) = \lambda^d f_i(x_0, \dots, x_n, a_0, \dots, a_m) = \lambda^d g_i(x_0, \dots, x_n)$.

$a \notin \pi(Z) \Leftrightarrow$ ne obstaja $x \in \mathbb{P}^n$, da je $(x, a) \in Z = V(f_1, \dots, f_r) \Leftrightarrow$ ne obstaja $x \in \mathbb{P}^n$, da za vsak $j = 1, \dots, r$ velja $f_j(x_0, \dots, x_n, a_0, \dots, a_m) = 0 \Leftrightarrow$ $\nexists x \in \mathbb{P}^n$, da za vse $j = 1, \dots, r$ velja $g_j(x) = 0 \Leftrightarrow V_p(g_1, \dots, g_n) = \emptyset \Leftrightarrow$ $\sqrt{(g_1, \dots, g_r)} = (1)$ ali $\sqrt{(g_1, \dots, g_r)} = (x_0, \dots, x_n)$ (projektivni Nullstellensatz) $\Leftrightarrow (x_0, \dots, x_n) \subseteq \sqrt{(g_1, \dots, g_r)} \Leftrightarrow \forall i = 0, \dots, n \, \exists k_i \in \mathbb{N}. \, x_i^{k_i} \in (g_1, \dots, g_r)$.

$(x_0, \dots, x_n)$ je maksimalen ideal

Označimo s $\mathbb{k}[x_0, \dots, x_n]_\ell$ prostor homogenih polinomov stopnje ℓ.

Dokažimo: $\forall i \, \exists k_i \in \mathbb{N}. \, x_i^{k_i} \in (g_1, \dots, g_n) \Leftrightarrow \exists \ell. \, \mathbb{k}[x_0, \dots, x_n]_\ell \subseteq (g_1, \dots, g_r)$.

($\Rightarrow$): Vzamemo $\ell = k_0 + k_1 + \dots + k_n$. Elementi $\mathbb{k}[x_0, \dots, x_n]_\ell$ so linearne kombinacije monomov $x_0^{p_0} x_1^{p_1} \cdots x_n^{p_n}$, kjer je $p_0 + \dots + p_n = \ell$.

$\Rightarrow p_i \geq k_i$ za vsaj en i

$$x_i^{p_i} \in (g_1, \ldots, g_r) \Rightarrow \underbrace{x_0^{p_0} \cdots x_n^{p_n} \in (g_1, \ldots, g_r)}_{(g_1, \ldots, g_r) \text{ je ideal}}$$

($\Leftarrow$): Za k_i lahko vzamemo l.

Dokazali smo: $a \notin \Pi(Z) \Leftrightarrow \exists l$, da je $\mathbb{K}[x_0, \ldots, x_n]_l \subseteq (g_1, \ldots, g_r)$.

$g_1, \ldots, g_r$ so homogeni iste stopnje d. Če je $\mathbb{K}[x_0, \ldots, x_n]_l \subseteq (g_1, \ldots, g_r)$, potem je $l \geq d$ in velja $\mathbb{K}[x_0, \ldots, x_n]_l \subseteq \underbrace{(g_1, \ldots, g_r)_l}_{\text{vsi homogeni polinomi stopnje } l \text{, ki pripadajo temu idealu}}$

Vedno velja $(g_1, \ldots, g_r)_l \subseteq \mathbb{K}[x_0, \ldots, x_n]_l$.

Sledi: $a \notin \Pi(Z) \Leftrightarrow \exists l \geq d$, da je $\mathbb{K}[x_0, \ldots, x_n]_l = (g_1, \ldots, g_r)_l$.

Po definiciji ideala generiranega z $g_1, \ldots, g_r$ je

$$(g_1, \ldots, g_r) = \{h_1 g_1 + \cdots + h_r g_r \mid h_1, \ldots, h_r \in \mathbb{K}[x_0, \ldots, x_n]\}.$$

$$\Rightarrow (g_1, \ldots, g_r)_l = \{h_1 g_1 + \cdots + h_r g_r \mid h_1, \ldots, h_r \in \mathbb{K}[x_0, \ldots, x_n]_{l-d}\}.$$

Definirajmo preslikavo

$$F_l : \left(\mathbb{K}[x_0, \ldots, x_n]_{l-d}\right)^r \longrightarrow \mathbb{K}[x_0, \ldots, x_n]_l$$
$$(h_1, \ldots, h_r) \longmapsto h_1 g_1 + \cdots + h_r g_r$$

Očitno velja: $\mathbb{K}[x_0, \ldots, x_n]_l = (g_1, \ldots, g_r)_l \Leftrightarrow F_l$ je surjektivna.

Sledi: $a \notin \Pi(Z) \Leftrightarrow \exists l \geq d$, da je F_l surjektivna.

F_l je linearna preslikava, zato ji lahko priredimo matriko glede na neki fiksni bazi prostorov $(\mathbb{K}[x_0, \ldots, x_n]_{l-d})^r$ in $\mathbb{K}[x_0, \ldots, x_n]_l$.

Matriko bomo tudi označili s F_l.

$\dim \mathbb{K}[x_0, \ldots, x_n]_l = \binom{n+l}{l}$

$\Rightarrow F_l$ ima $r \cdot \binom{n+l-d}{l-d}$ stolpcev in $\binom{n+l}{l}$ vrstic

$\Rightarrow F_l$ je surjektivna $\Leftrightarrow \operatorname{rang} F_l = \binom{n+l}{l} \Leftrightarrow \operatorname{rang} F_l \geq \binom{n+l}{l}$

Dokazali smo: $a \notin \Pi(Z) \Leftrightarrow \exists l \geq d$, da je $\operatorname{rang} F_l = \binom{n+l}{l} \Leftrightarrow \exists l \geq d$, da je vsaj en minor reda $\binom{n+l}{l}$ matrike F_l neničeln.

Matrika F_l ima za člene koeficiente polinomov $g_1, \ldots, g_r$.

Koeficienti polinomov $g_1, \ldots, g_r$ so polinomi v homogenih koordinatah točke a, in to so homogeni polinomi stopnje d v homogenih koordinatah. Členi matrike F_ℓ so torej homogeni polinomi stopnje d v homogenih koordinatah točke a. Njeni minorji so torej homogeni polinomi v homogenih koordinatah točke a. Pogoj, da je nek minor enak 0, je torej pogoj, da a leži na neki raznoterosti (= zaprt pogoj). Pogoj, da obstaja nek minor, ki ni enak 0, je odprt pogoj.

Pokazali smo, da $a \notin \Pi(Z) \iff$ obstaja $\ell \geq d$ in obstaja minor reda $\binom{n+\ell}{\ell}$ matrike F_ℓ, ki ni 0. Torej je $a \notin \Pi(Z) \iff$ a pripada neki odprti podmnožici v $\mathbb{P}^m$.

$\Rightarrow a \in \Pi(Z) \iff a$ pripada neki zaprti podmnožici v $\mathbb{P}^m$.

$\Rightarrow \Pi(Z)$ je projektivna raznoterost v $\mathbb{P}^m$. $\square$

Raznoterosti, ki so podane z minorji neke matrike polinomov, se imenujejo determinantne raznoterosti.

Isti dokaz pokaže:

Posledica: Projekcija $\mathbb{P}^n \times \mathbb{A}^m \longrightarrow \mathbb{A}^m$ je zaprta.

Kaj je drugače v dokazu?

Če je Z zaprta v $\mathbb{P}^n \times \mathbb{A}^n$, je $Z = V(F_1, \dots, F_r)$, kjer so $F_i \in \mathbb{k}[x_0, \dots, x_n, y_1, \dots, y_m]$ homogeni v $x_0, \dots, x_n$, ne pa nujno v $y_1, \dots, y_n$. Lahko predpostavimo, da so homogeni iste stopnje v $x_0, \dots, x_n$. Za $a = (a_1, \dots, a_m) \in \mathbb{A}^m$ definiramo polinome g_i enako kot v prejšnjem dokazu. Ti so stopnje d. Nato je dokaz enak in dobimo $a \notin \Pi(Z) \iff \exists \ell \geq d$ in obstaja minor reda $\binom{n+\ell}{\ell}$ matrike F_ℓ, ki ni 0. Edina razlika je v tem, da minorji niso homogeni polinomi, zato je $\Pi(Z)$ afina raznoterost v $\mathbb{A}^m$.

Opomba: Naj bo $Z = V(f_1, \dots, f_r)$ kot v dokazu. Potem je $y = (y_0 : \dots : y_m) \in \pi(Z) \Leftrightarrow \exists x = (x_0 : \dots : x_n) \in \mathbb{P}^n$, da je $(x, y) \in Z$, torej $f_i(x_0, \dots, x_n, y_0, \dots, y_m) = 0$ za vsak i. Sliko $\pi(Z)$ torej dobimo tako, da iz enačb $f_i(x, y) = 0$ eliminiramo $x_0, \dots, x_n$. Zato v računski algebraični geometriji temu izreku običajno rečejo osnovni izrek eliminacijske teorije.

Posledica: Naj bo X afina raznoterost in Y projektivna raznoterost. Potem je projekcija $Y \times X \longrightarrow X$ zaprta.

Dokaz: $Y \times X$ je zaprta podmnožica v $\mathbb{P}^n \times X$, zato, če je Z zaprta v $Y \times X$, je zaprta tudi v $\mathbb{P}^n \times X$. Za Y lahko torej brez škode za splošnost vzamemo $\mathbb{P}^n$. Naj bo Z zaprta v $\mathbb{P}^n \times X$. X je afina raznoterost, zato je X zaprta v $\mathbb{A}^m$ za nek $m \Rightarrow Z$ je zaprta v $\mathbb{A}^m$. Zato je $\pi(Z)$ zaprta v $\mathbb{A}^m$ in zato tudi v X.

Izrek: Naj bo X kvaziprojektivna raznoterost in Y projektivna raznoterost. Potem je projekcija $\pi: Y \times X \longrightarrow X$ zaprta.
„Projekcija vzdolž projektivne raznoterosti je zaprta."

Dokaz: Kot v prejšnji posledici lahko predpostavimo, da je $Y = \mathbb{P}^n$. Zadnjič smo pokazali, da ima vsaka točka vsake kvaziprojektivne raznoterosti okolico, ki je izomorfna afini raznoterosti. Obstaja torej pokritje $X = \bigcup_{i \in I} U_i$, kjer je vsak U_i izomorfen neki afini raznoterosti $Z_i \subseteq \mathbb{A}^{n_i}$.
Naj bo $W \subseteq \underbrace{\mathbb{P}^n}_{=Y} \times X$ poljubna zaprta množica. Za vsak $i \in I$ se zožitev $\pi|_{\mathbb{P}^n \times U_i}$ faktorizira kot

$$\mathbb{P}^n \times U_i \longrightarrow \mathbb{P}^n \times \underset{\mathbb{A}^{n_i}}{\underset{\cap}{Z_i}} \to \underset{\mathbb{A}^{n_i}}{\underset{\cap}{Z_i}} \to U_i$$

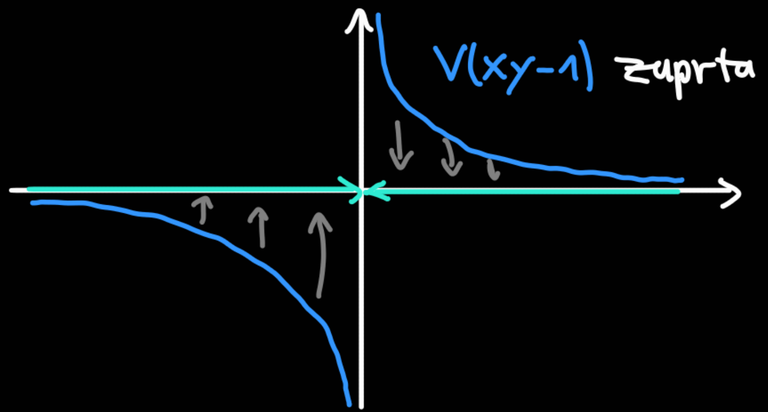
Srednja preslikava je zaprta po prejšnji posledici, ostali preslikavi sta izomorfizma, zato je kompozitum zaprta preslikava. $\Rightarrow$ $\underbrace{\pi(W \cap (\mathbb{P}^n \cap U_i))}_{\pi(W) \cap U_i}$ je zaprta v U_i za vsak i.

Zaprtost je lokalna lastnost $\Rightarrow$ $\pi(W)$ je zaprta v X. $\square$

<u>Definicija:</u> Kvaziprojektivna raznoterost Y, za katero je projekcija $Y \times X \longrightarrow X$ za vsako kvaziprojektivno raznoterost X, se imenuje **polna** raznoterost. (angleško: complete)

Pokazali smo, da so projektivne raznoterosti polne.

<u>Primer:</u> $\mathbb{A}^1$ ni polna raznoterost



<u>Opomba:</u> Obstajajo polne raznoterosti, ki niso projektivne.

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<u>Definition:</u> Let X be a quasiprojective variety. The set $\Delta_X = \{(x,x) \in X \times X\}$ is called the **diagonal** of X.

<u>Proposition:</u> The diagonal Δ_X is closed in $X \times X$.

<u>Proof:</u> Assume that X is an open subset of a closed subset of $\mathbb{P}^n$. Then $\Delta_X = \Delta_{\mathbb{P}^n} \cap (X \times X)$.
Since the Zariski topology on subsets of $\mathbb{P}^n \times \mathbb{P}^n$ is the

relative topology, it is enough to show that $\Delta_{\mathbb{P}^n}$ is closed in $\mathbb{P}^n \times \mathbb{P}^n$.
$\Delta_{\mathbb{P}^n}$ is indeed closed in $\mathbb{P}^n \times \mathbb{P}^n$, as $V(x_i y_j - x_j x_i \mid i,j = 0,\dots,n)$.
If $(a,b) = ((a_0:\dots:a_n),(b_0:\dots:b_n)) \in \mathbb{P}^n \times \mathbb{P}^n$ is such that $a_i b_j = a_j b_i \ \forall i,j$ and $a_i = 0$, then $b_j = \frac{a_j b_i}{a_i} \ \forall j$

$\Rightarrow b = (b_0:\dots:b_n) = \left(\frac{a_0 b_i}{a_i} : \dots : \frac{a_n b_i}{a_i} \right) = \frac{b_i}{a_i}(a_0:\dots:a_n) = (a_0:\dots:a_n) = a$

$\Rightarrow (a,b) \in \mathbb{P}^n \times \mathbb{P}^n$ ▨

<u>Definition:</u> Let X, Y be quasiprojective varieties and $\phi: X \longrightarrow Y$ a regular map. The graph of ϕ is the set $\Gamma_\phi = \{(x, \phi(x)) \mid x \in X\} \subseteq X \times Y$.

<u>Proposition:</u> The graph of a regular map $\phi: X \to Y$ is closed in $X \times Y$.

<u>Proof:</u> Define the map
$$\overset{\phi \times id_X}{\overset{\shortparallel}{\psi}} : X \times Y \longrightarrow Y \times Y$$
$$(x,y) \longmapsto (\phi(x), y).$$
<u>Exercise:</u> This is a regular map.
$\Rightarrow$ It is continuous in the Zariski topology.
$$\Gamma_\phi = \{(x,y) \in X \times Y \mid \phi(x) = y\}$$
$$= \{(x,y) \in X \times Y \mid \psi(x,y) = (y,y)\}$$
$$= \psi^{-1}(\Delta_Y)$$
Δ_Y is closed in $Y \times Y$, ψ is continuous, so Γ_ϕ is closed in $X \times Y$. ▨

<u>Theorem:</u> Let X be a projective variety, Y a quasi-projective variety and $\phi: X \to Y$ a regular map. Then ϕ is a closed map.

<u>Proof</u>: Closed subsets of X are again projective varieties, so it is enough to show that $\phi(X)$ is closed.

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & Y \\ & \Gamma_\phi \subseteq X \times Y \arrow[ur, "\pi_2"'] & \end{tikzcd}$$

$$x \longmapsto (x, \phi(x)) \longmapsto \phi(x)$$

$\phi(X) = \pi_2(\Gamma_\phi)$, where $\pi_2 : X \times Y \longrightarrow Y$ is the projection to the second factor.

By the proposition, the graph Γ_ϕ is closed in $X \times Y$. π_2 is a projection along a projective variety, so it is closed by a theorem from last time. $\phi(X) = \pi_2(\Gamma_\phi)$ is closed in Y. $\blacksquare$

Corollary: Let X be an irreducible projective variety and ϕ a regular function on X ($\phi : X \longrightarrow \mathbb{k}$, $\phi \in \mathcal{O}_X(X)$). Then ϕ is constant.

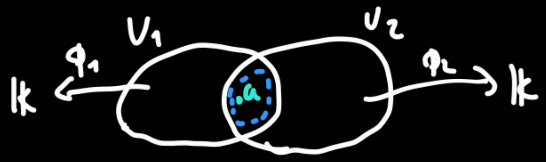
<u>Proof</u>: We can view ϕ as a map $\phi : X \longrightarrow \mathbb{A}^1$. We can also view it as a map $\Phi : X \longrightarrow \mathbb{P}^1$. The image of Φ is not the whole $\mathbb{P}^1$. By the theorem, the image of Φ is closed in $\mathbb{P}^1$. The image is not $\mathbb{P}^1$, so it has to be a finite set. X is irreducible, so the image is one point. $\blacksquare$

Corollary: Let X be an irreducible projective variety, $Y \subseteq \mathbb{A}^m$ an affine variety and $\phi : X \longrightarrow Y$ a regular map. Then ϕ is constant.

<u>Proof</u>: ϕ is given by an n-tuple of regular functions, which are all constant.

4. Germs of regular functions (zarodki regularnih funkcij)

<u>Definition</u>: Let X be a quasiprojective variety and $a \in X$ a point. For open neighbourhoods U_1 and U_2 of a in X and regular functions $\phi_1 \in \mathcal{O}_X(U_1)$, $\phi_2 \in \mathcal{O}_X(U_2)$ we define $(U_1, \phi_1) \sim (U_2, \phi_2) :\Leftrightarrow$ there exists an open neighbourhood $U \subseteq U_1 \cap U_2$ of a such that $\phi_1|_U = \phi_2|_U$.



This is an equivalence relation (exercise).

The quotient set

$$\{(U, \phi) \mid U \text{ open neighbourhood of } a,\ \phi \in \mathcal{O}_X(U)\}/_\sim$$

is denoted by $\mathcal{O}_{X,a}$.

The elements of $\mathcal{O}_{X,a}$ are called germs of regular functions in the point a.

On $\mathcal{O}_{X,a}$ we can define addition, multiplication, and multiplication with scalars. We define addition and multiplication as follows: If U_1, U_2 are open neighbourhoods of a, $\phi_1 \in \mathcal{O}_X(U_1)$, $\phi_2 \in \mathcal{O}_X(U_2)$, then on $U_1 \cap U_2$ we can define $\phi_1 + \phi_2$ and $\phi_1 \cdot \phi_2$, and these are regular functions.

Define:
$$[(U_1, \phi_1)] + [(U_2, \phi_2)] = [(U_1 \cap U_2, \phi_1|_{U_1 \cap U_2} + \phi_2|_{U_1 \cap U_2})],$$
$$[(U_1, \phi_1)] \cdot [(U_2, \phi_2)] = [(U_1 \cap U_2, \phi_1|_{U_1 \cap U_2} \cdot \phi_2|_{U_1 \cap U_2})].$$

<u>Exercise</u>: The operations are well defined and $\mathcal{O}_{X,a}$ is a $\mathbb{k}$-algebra for these operations.

<u>Definition</u>: $\mathcal{O}_{X,a}$ is called the local ring of X in a or the ring of germs of regular functions in $a \in X$.

Trditev: Naj bo X afina raznoterost in $a \in X$. Potem je $\mathcal{O}_{X,a}$ izomorfna lokalizaciji $\mathbb{k}[X]_{M_a}$, kjer je $M_a = \{f \in \mathbb{k}[X] \mid f(a) = 0\}$ maksimalen ideal v $\mathbb{k}[X]$.

December 2, 2025

Dokaz: Imamo homomorfizem algeber $\mathbb{k}[X] \longrightarrow \mathbb{k}$, $f \longmapsto f(a)$ z jedrom M_a. Ta homomorfizem je surjektiven, zato je $\mathbb{k} \cong \mathbb{k}/M_a$. Ker je $\mathbb{k}$ polje, sledi, da je M_a maksimalen ideal.

Definiramo preslikavo $F: \mathbb{k}[X]_{M_a} \longrightarrow \mathcal{O}_{X,a}$, $\frac{f}{g} \longmapsto [(D(g), \tau)]$, kjer je $\tau(x) = \frac{f(x)}{g(x)}$ za $x \in D(g)$. $g \notin M_a \Rightarrow g(a) \neq 0 \Rightarrow a \in D_g$: $D(g)$ je res okolica točke a, τ je očitno regularna funkcija na $D(g)$.

$\Rightarrow [(D(g), \tau)]$ je res element $\mathcal{O}_{X,a}$.

Dobra definiranost preslikave F: Recimo, da je $\frac{f}{g} = \frac{f'}{g'}$ v $\mathbb{k}[X]_{M_a}$. To pomeni, da obstaja $h \in \mathbb{k}[X] \setminus M_a$, da je $h(fg' - f'g) = 0$ v $\mathbb{k}[X]$.

$\Rightarrow h(x)(f(x)g'(x) - f'(x)g(x)) = 0 \quad \forall x \in X$

$h \notin M_a \Rightarrow h(a) \neq 0 \Rightarrow a \in D(h) \Rightarrow D(h) \cap D(g) \cap D(g') = U$ je odprta okolica točke a, ki je vsebovana v $D(g) \cap D(g')$.

Za $x \in U$ je $f(x) \cdot g'(x) - f'(x) g(x) = 0 \Rightarrow \frac{f(x)}{g(x)} = \frac{f'(x)}{g'(x)}$.

$\Rightarrow (D(g), \frac{f}{g}) \sim (D(g'), \frac{f'}{g'}) \Rightarrow F$ je dobro definirana.

Preverimo lahko, da je F homomorfizem algeber (DN).

Injektivnost: $\frac{f}{g} \in \ker F \Rightarrow (D(g), \frac{f}{g}) \sim (X, 0)$.

To pomeni, da obstaja odprta okolica U točke a v $D(g)$, da je $f(x) = 0$ za vse $x \in U$. To enakost lahko gledamo na poljubni odprti podmnožici v U, zato lahko predpostavimo, da je U odlikovana odprta podmnožica v X, torej $U = D(h)$ za nek $h \in \mathbb{k}[X]$. $\Rightarrow h(x) f(x) = 0 \quad \forall x \in X$

$$h(f(x) 1 - 0\, g(x)) = 0 \quad \forall x \in X$$
$$\Rightarrow h(f \cdot 1 - 0 \cdot g) = 0 \quad \text{v } \mathbb{k}[X]$$
$$\Rightarrow \frac{f}{g} = \frac{0}{1} \quad \text{v } \mathbb{k}[X]_{M_a}$$
$$\Rightarrow F \text{ je injektivna}$$

Surjektivnost: Naj bo $[(U, \tau)] \in \mathcal{O}_{X,a}$ poljuben. Po definiciji regularne funkcije obstaja odprta okolica U_a za a v U (ki je odprta tudi v X) in obstajata $f_a, g_a \in \mathbb{k}[X]$, da za $x \in U_a$ velja $g_a(x) \neq 0$ in $\varphi(x) = \frac{f_a(x)}{g_a(x)}$.

$\Rightarrow (U, \varphi) \underset{\uparrow}{\sim} (D(g_a), \frac{f_a}{g_a}) \Rightarrow [(U, \varphi)] = F(\frac{f_a}{g_a})$. ▨

$V_a \subseteq U \cap D(g_a)$

Posledica: Naj bo X afina raznoterost in $a \in X$. Potem je $\mathcal{O}_{X,a}$ lokalen kolobar z edinim maksimalnim idealom $M_{X,a} = \{[(U, \varphi)] \mid \varphi(a) = 0\}$.

Posledica: Naj bo X kvaziprojektivna raznoterost in $a \in X$. Potem je $\mathcal{O}_{X,a}$ lokalen kolobar z edinim maksimalnim idealom $M_{X,a} = \{[(U, \varphi)] \mid \varphi(a) = 0\}$.

Dokaz: Vemo, da ima a neko odprto okolico V, ki je izomorfna afini raznoterosti. Če je $[(U, \varphi)]$ ekvivalenčni razred v $\mathcal{O}_{X,a}$, potem je $(U, \varphi) \sim (U \cap V, \varphi|_{U \cap V}) \Rightarrow \mathcal{O}_{X,a} \cong \mathcal{O}_{V,a}$. Sedaj upoštevamo prejšnjo posledico. ▨

Spomnimo se, da množice $\mathcal{O}_X(V)$, kjer je V odprta podmnožica v X, tvorijo snop kolobarjev na X.

Definicija: Naj bo $\mathcal{F}$ (pred)snop na topološkem prostoru X in $a \in X$. Za odprti podmnožici $U_1, U_2 \subseteq X$, ki vsebujeta a, in prereza $f_1 \in \mathcal{F}(U_1)$, $f_2 \in \mathcal{F}(U_2)$ definiramo:

$(U_1, f_1) \sim (U_2, f_2) :\Leftrightarrow$ obstaja odprta množica $U \subseteq U_1 \cap U_2$, ki vsebuje a, da je $res_{U_1|U}(f_1) = res_{U_2|U}(f_2)$.

To je ekvivalenčna relacija na parih (U, f), kjer je U odprta okolica za a v X in $f \in \mathcal{F}(U)$.

Kvocientna množica se imenuje bilka (ang. stalk) (pred)snopa

$\mathcal{F}$ v točki a. Oznaka $\mathcal{F}_a$.
Ekvivalenčnim razredom rečemo zarodki snopa $\mathcal{F}$.

Lokalni kolobar $\mathcal{O}_{X,a}$ je torej bilka strukturnega snopa $\mathcal{O}_X$.

Če je $\mathcal{F}$ snop Abelovih grup definiramo strukturo Abelove grupe na $\mathcal{F}_a$: Če sta $[(U_1,f_1)], [(U_2,f_2)] \in \mathcal{F}_a$, potem je $U_1 \cap U_2$ odprta okolica za a in lahko izračunamo
$\underbrace{res_{U_1|U_1\cap U_2}(f_1)}_{\in \mathcal{F}(U_1\cap U_2)} + \underbrace{res_{U_2|U_1\cap U_2}(f_2)}_{\in \mathcal{F}(U_1\cap U_2)}$.

Definiramo $[(U_1,f_1)] + [(U_2,f_2)] = [(U_1\cap U_2), res_{U_1|U_1\cap U_2}(f_1) + res_{U_2|U_1\cap U_2}(f_2)]$.
To je dobro definirano seštevanje in $(\mathcal{F}_a, +)$ je Abelova grupa.
Podobno: Če je $\mathcal{F}$ snop kolobarjev, je $\mathcal{F}_a$ kolobar.

5. Racionalne preslikave

Definicija: Naj bosta X in Y kvaziprojektivni raznoterosti. Racionalna preslikava iz X v Y je regularna preslikava $\phi: U \longrightarrow Y$, kjer je U odprta in gosta podmnožica v X in velja: Ne obstaja nobena regularna preslikava na odprti podmnožici v X, ki strogo vsebuje U in je razširitev ϕ. Pišemo $\phi: X \dashrightarrow Y$ (s tem poudarimo, da ϕ morda ni definirana na vsem X.
Racionalna preslikava $X \longrightarrow \mathbb{k}$ se imenuje racionalna funkcija. Množico racionalnih funkcij na X označimo s $\mathbb{k}(X)$.

Opomba: Če je X nerazcepna, je v definiciji dovolj predpostaviti, da je U odprta in neprazna. Potem bo U tudi gosta.

Opomba: Ekvivalentna definicija racionalne preslikave: Naj bosta U_1, U_2 odprti gosti podmnožici v X in $\phi_1 : U_1 \longrightarrow Y$, $\phi_2 : U_2 \to Y$ regularni preslikavi. Definiramo $\phi_1 \sim \phi_2 :\Leftrightarrow \phi_1|_{U_1 \cap U_2} = \phi_2|_{\underbrace{U_1 \cup U_2}_{\neq \emptyset,}}$ ker sta U_1 in U_2 gosti

Ekvivalenčnim razredom rečemo racionalne preslikave.

Primer [racionalna preslikava, ki ni regularna]:

$$\phi : \mathbb{A}^2 \dashrightarrow \mathbb{P}^1$$
$$(x,y) \longmapsto (x:y)$$

ϕ je regularna na $U = \mathbb{A}^2 \setminus \{(0,0)\}$, U je odprta in gosta v $\mathbb{A}^2$. Preverimo, da je ϕ res racionalna preslikava, kar je v tem primeru ekvivalentno temu, da ϕ ni mogoče razširiti na cel $\mathbb{A}^2$. Recimo, da je $\tilde{\Phi} : \mathbb{A}^2 \longrightarrow \mathbb{P}^1$ regularna razširitev ϕ-ja. Po definiciji potem obstaja odprta okolica U točke $(0,0)$ in obstajata polinoma $f, g \in \mathbb{k}[x,y]$, da je $\tilde{\Phi}(x,y) = (f(x,y) : g(x,y))$ za vse $(x,y) \in U$. Na $U \cap (\mathbb{A}^2 \setminus \{0,0\})$ velja $(f(x,y) : g(x,y)) = (x : y)$ $\Rightarrow x g(x,y) = y f(x,y)$. Presek $U \cap (\mathbb{A}^2 \setminus \{0,0\})$ je gost v $\mathbb{A}^2$ $\Rightarrow x g(x,y) = y f(x,y)$ je enakost v $\mathbb{k}[x,y]$. Ker je $\mathbb{k}[x,y]$ kolobar z enolično faktorizacijo, obstaja $h \in \mathbb{k}[x,y]$, da je $f(x,y) = x h(x,y)$ in $g(x,y) = y h(x,y)$ $\Rightarrow \tilde{\Phi}(x,y) = (x h(x,y) : y h(x,y))$ $\forall (x,y) \in U$, $\tilde{\Phi}(0,0) = (0,0)$ $\times$ protislovje

Racionalna preslikava je poseben primer regularne preslikave, zato je zvezna, kjer je definirana.

Posledica: Če je X nerazcepna kvaziprojektivna raznoterost in $\phi : X \dashrightarrow Y$ racionalna preslikava, potem je $\overline{\phi(X)}$ nerazcepna.

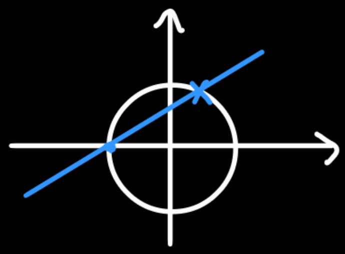
Posledica: Če je $\phi : \mathbb{A}^n \dashrightarrow X$ racionalna preslikava, je $\overline{\phi(\mathbb{A}^n)}$ nerazcepna.

Definicija: Dominantni racionalni preslikavi $\phi : \mathbb{A}^n \dashrightarrow X$ pravimo racionalna parametrizacija raznoterosti X.

Raznoterosti, ki imajo racionalno parametrizacijo, so torej nerazcepne. (Pogost način za dokazovanje nerazcepnosti.)

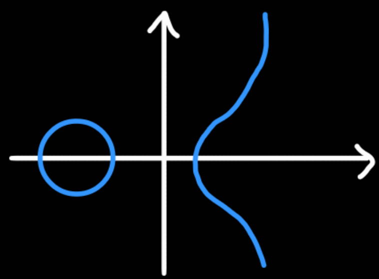
Primer: Racionalna parametrizacija krožnice.

$V(x^2+y^2-1)$:



$$\phi(t) = \left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right)$$

Primer: Gladka kubična ravninska krivulja nima racionalne parametrizacije.



Lemma: Naj bo U odprta in gosta podmnožica kvaziprojektivne raznoterosti X in $\phi : U \longrightarrow \mathbb{k}$ regularna preslikava. Potem obstaja natanko ena racionalna preslikava $X \dashrightarrow Y$, ki razširja ϕ.

Dokaz: Racionalna razširitev ϕ-ja obstaja po Zornovi lemi (natančen razmislek DN).

Enoličnost: Recimo, da sta $\tau_1 : V_1 \longrightarrow \mathbb{k}$ in $\tau_2 : V_2 \longrightarrow \mathbb{k}$ racionalni preslikavi $X \dashrightarrow \mathbb{k}$, ki razširjata ϕ. Na $V_1 \cap V_2$ lahko definiramo regularno funkcijo $\tau_1 - \tau_2 : V_1 \cap V_2 \longrightarrow \mathbb{k}$. Vemo, da je $V(\tau_1 - \tau_2)$ zaprta v $V_1 \cap V_2$.

$\{x \in V_1 \cap V_2 ;\ \tau_1(x) = \tau_2(x)\}$

zaprtje v $V_1 \cap V_2$

τ_1 in τ_2 se ujemata na $U \Rightarrow U \subseteq V(\tau_1 - \tau_2) \Rightarrow \overline{U} \subseteq V(\tau_1 - \tau_2)$

U je gosta $\Rightarrow V(\tau_1 - \tau_2) = V_1 \cap V_2 \Rightarrow \tau_1(x) = \tau_2(x)$ za $x \in V_1 \cap V_2$.
$\Rightarrow$ Lahko definiramo racionalno preslikavo $\tau : V_1 \cup V_2 \longrightarrow \mathbb{k}$

$$\tau(x) = \begin{cases} \tau_1(x); & x \in V_1 \\ \tau_2(x); & x \in V_2 \end{cases}$$

τ je razširitev τ_1 in τ_2. Zaradi maksimalnosti je $V_1 \cup V_2 = V_1 = V_2$ in $\tau_1 = \tau_2$. ▨

Ista lema velja za racionalne preslikave $X \dashrightarrow Y$.

Na $\mathbb{k}(X)$ definiramo množenje s skalarji na očiten način.
Seštevanje in množenje: Naj bosta $\phi_1 : U_1 \longrightarrow \mathbb{k}$ in $\phi_2 : U_2 \longrightarrow \mathbb{k}$ racionalni funkciji $X \dashrightarrow \mathbb{k}$. U_1 in U_2 sta odprti in gosti. Njun presek je odprt in gost v X in na preseku definiramo $\phi_1 + \phi_2 : U_1 \cap U_2 \longrightarrow \mathbb{k}$ in $\phi_1 \cdot \phi_2 : U_1 \cap U_2 \longrightarrow \mathbb{k}$.
Po lemi obstajata enolični razširitvi teh dveh regularnih funkcij. Definiramo, da sta ti dve razširitvi vsota in produkt racionalnih funkcij $\phi_1, \phi_2 : X \dashrightarrow \mathbb{k}$.

5. december 2025

Lema: $\mathbb{k}(X)$ je $\mathbb{k}$-algebra.

Izrek: Naj bo X nerazcepna afina raznoterost. Potem je $\mathbb{k}(X)$ polje, izomorfno polju ulomkov kolobarja $\mathbb{k}[X]$.

Dokaz: F naj bo polje ulomkov kolobarja $\mathbb{k}[X]$. Definirajmo preslikavo $\Phi : F \longrightarrow \mathbb{k}(X)$ na naslednji način: Naj bo $\frac{f}{g} \in F$.
Potem je $D(g)$ odprta podmnožica v X, ki je tudi gosta v X, saj je X nerazcepna. Na $D(g)$ lahko definiramo regularno funkcijo $x \longmapsto \frac{f(x)}{g(x)}$. To regularno funkcijo enolično razširimo do racionalne funkcije na X in definiramo, da je ta racionalna funkcija $\Phi(\frac{f}{g})$.
To je homomorfizem algeber (DN).

Injektivnost: Naj bo $\frac{f}{g} \in \ker \Phi$. Potem je $f(x)=0$ za vse $x \in D(g)$. $D(g)$ je gosta, torej je $f(x)=0$ za vse $x \in X$ oziroma $f=0$ v kolobarju $\Bbbk[X] \Rightarrow \ker \Phi$ je trivialno.

Surjektivnost: Naj bo $\varphi \in \Bbbk(X)$ poljubna. Potem je $\varphi : U \longrightarrow \Bbbk$ regularna funkcija za neko odprto in gosto podmnožico U. Naj bo $a \in U$. Potem obstaja odprta okolica V_a za a v U in obstajata $f_a, g_a \in \Bbbk[X]$, da je $\frac{f_a(x)}{g_a(x)} = \varphi(x)$ za vse $x \in V_a$. V_a je odprta v X in tudi gosta, saj je X nerazcepna. Predpostavimo lahko, da je $V_a = D(g_a)$. φ je potem racionalna funkcija, ki je razširitev regularne funkcije $D(g_a) \longrightarrow \Bbbk$, $x \longmapsto \frac{f_a(x)}{g_a(x)}$, torej je $\varphi = \Phi\left(\frac{f_a}{g_a}\right)$. ▨

Če je R cel kolobar, $M \triangleleft R$ maksimalen ideal in F polje ulomkov kolobarja R, potem je R_M izomorfen kolobarju vseh ulomkov v F, ki so oblike $\frac{a}{b}$, kjer $b \notin M$.
To dejstvo uporabimo za $R = \Bbbk[X]$, kjer je X nerazcepna afina raznoterost, $F = \Bbbk(X)$ in $M = M_a = \{f \in \Bbbk[X] \mid f(a) = 0\}$.

Posledica: Naj bo X nerazcepna afina raznoterost in $a \in X$. Potem je $\mathcal{O}_{X,a} \cong \{\varphi : U \longrightarrow \Bbbk \text{ racionalna funkcija} \mid a \in U, U \text{ odprta v } X\}$. To je torej kolobar vseh racionalnih funkcij, ki so definirane v a.

Lema: Naj bo X nerazcepna kvaziprojektivna raznoterost in $V \neq \emptyset$ njena odprta podmnožica. Potem je $\Bbbk(X) \cong \Bbbk(V)$.

Dokaz: $\Bbbk(X) \underset{G}{\overset{F}{\rightleftarrows}} \Bbbk(V)$
Če je $\varphi : X \dashrightarrow \Bbbk$ racionalna funkcija z definicijskim območjem V, potem je $\varphi|_{U \cap V}$ racionalna funkcija na U. Definiramo $F(\varphi) := \varphi|_{U \cap V}$.

Če je $f: U \dashrightarrow \mathbb{k}$ racionalna funkcija, je $f: W \longrightarrow \mathbb{k}$ regularna funkcija za neko odprto in gosto podmnožico W v U, ki je tudi odprta in gosta v X. Vemo, da se f enolično razširi do racionalne funkcije $\tilde{f}: X \dashrightarrow \mathbb{k}$. Definiramo $G(f) := \tilde{f}$.

Preverimo lahko, da sta F in G homomorfizma, ki sta drug drugemu inverzna. ▨

Posledica: Če je X nerazcepna projektivna raznoterost, je $\mathbb{k}(X)$ polje.

$(\mathbb{k}(X) \cong \mathbb{k}(X \cap U_0))$

S kompozitumom racionalnih funkcij je težava:

Primer:

$$\begin{aligned} \phi: \mathbb{A}^1 &\longrightarrow \mathbb{A}^2 \\ x &\longmapsto (x, 0) \end{aligned}$$

regularna preslikava na $\mathbb{A}^1$ $\Rightarrow$ racionalna

$$\begin{aligned} f: \mathbb{A}^2 &\dashrightarrow \mathbb{A}^1 \\ (x,y) &\longmapsto \frac{x}{y} \end{aligned}$$

to je tudi racionalna preslikava

$\phi(\mathbb{A}^1) = V(y)$, kar ima prazen presek z definicijskim območjem $f \Rightarrow$ ne moremo definirati kompozituma $f \circ \phi$.

Teh težav nimamo, če obravnavamo dominantne racionalne preslikave, torej tiste, kjer je slika gosta.

Definicija: Naj bosta $\phi_1: X_1 \dashrightarrow X_2$ in $\phi_2: X_2 \dashrightarrow X_3$ racionalni preslikavi med nerazcepnima kvaziprojektivnima raznoterostima. Dodatno predpostavimo, da je ϕ_1 dominantna. To pomeni: $\overline{\phi_1(X_1)} = X_2$. Naj bo U_1 definicijsko območje ϕ_1 in U_2 definicijsko območje ϕ_2. Po Chevalleyevem izreku je $\phi_1(X_1)$ konstruktibilna množica, torej je oblike $\phi_1(X_1) = Z_1 \setminus (Z_2 \setminus (Z_3 \setminus \cdots) \cdots)$, kjer je $Z_1 \supseteq Z_2 \supseteq \cdots$ padajoče zaporedje zaprtih množic v X_2. $\Phi(X_1) = Z_1 \Rightarrow Z_1 = X_2$. $X_2 \setminus Z_2$ je odprta in vsebovana v $X_2\setminus(Z_2\setminus(Z_3\setminus\cdots))$ $\Rightarrow X_2\setminus Z_2$ je vsebovana v $\Phi_1(X_1)$ in je odprta v X_2 in gosta. U_2 je tudi odprta in gosta v $X_2 \Rightarrow U_2\cap(X_2\setminus Z_2)$ je odprta in gosta v Z_2. Φ_1 je zvezna $\Rightarrow \Phi_1^{-1}(U_2\cap(X_2\setminus Z_2))$ je odprta v X_1. Je tudi gosta, ker je X_1 nerazcepna. Na $\Phi_1^{-1}(U_1\cap(X_2\setminus Z_2))$ lahko definiramo regularno preslikavo $\Phi_2\circ\Phi_1$. To lahko enolično razširimo do racionalne preslikave $X_1 \dashrightarrow X_3$. Tej racionalni preslikavi rečemo **kompozitum** $\Phi_2\circ\Phi_1$.

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<u>Definicija</u>: Naj bosta X in Y kvaziprojektivni nerazcepni raznoterosti. Dominantna racionalna preslikava $\Phi: X \dashrightarrow Y$ je **biracionalna** ali **biracionalna ekvivalenca**, če obstaja dominantna racionalna preslikava $\Psi: Y \dashrightarrow X$, da je $\Phi\circ\Psi = id_Y$ in $\Psi\circ\Phi = id_X$ (pri čemer kompozitum definiramo kot zadnjič). V tem primeru pravimo, da sta X in Y **biracionalni** ali **biracionalno ekvivalentni**. Kvaziprojektivna raznoterost je **racionalna**, če je biracionalno ekvivalentna $\mathbb{A}^n$ za nek n.

<u>Primer</u>: $V(x^2+y^2-1)$ je racionalna, biracionalno je ekvivalentna $\mathbb{A}^1$:

$$\begin{array}{ccc} \mathbb{A}^1 & \substack{\dashrightarrow \\ \dashleftarrow} & V(x^2+y^2-1) \\ t & \longmapsto & \left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right) \\ \frac{y}{1+x} & \longleftarrow & (x,y) \end{array}$$

<u>Opomba</u>: Izomorfni raznoterosti sta vedno biracionalno ekvivalentni. Obratno ne velja nujno.

<u>Primer</u>:
$$\begin{array}{ccc} \mathbb{A}^1 & \substack{\longrightarrow \\ \dashleftarrow} & V(x^2-y^3) \\ t & \longmapsto & (t^3, t^2) \\ \frac{x}{y} & \longleftarrow & (x,y) \end{array}$$

To sta biracionalni ekvivalenci, toda $V(x^2-y^3)$ ni izomorfno $\mathbb{A}^1$.

Trditev: Nerazcepni kvaziprojektivni raznoterosti X_1 in X_2 sta biracionalno ekvivalentni $\Leftrightarrow$ obstajata neprazni odprti podmnožici $U_1 \subseteq X_1$ in $U_2 \subseteq X_2$, ki sta izomorfni.

Dokaz: ($\Rightarrow$): Naj bosta $\phi: X_1 \dashrightarrow X_2$ in $\psi: X_2 \dashrightarrow X_1$ biracionalni ekvivalenci: $\phi \circ \psi = id$, $\psi \circ \phi = id$. ϕ naj bo definirana na V_1, ψ pa na V_2, kjer sta V_1, V_2 odprti množici.
ϕ je dominantna, zato obstaja odprta množica U v X_2, da je $U \subseteq \phi(X_1)$. Definiramo $U_2 = U \cap V_2$. Ta množica je odprta in gosta. Definiramo $U_1 = \phi^{-1}(U_2)$, ki je odprta in gosta v X_1. Za $x \in U_1$ je definiran $\psi \circ \phi$, ki je po predpostavki identiteta. Na U_2 pa je definiran kompozitum $\phi \circ \psi$, ki je po predpostavki tudi enak identiteti $\Rightarrow U_1$ in U_2 sta izomorfni.
($\Leftarrow$): Recimo, da imamo izomorfizma $U_1 \underset{\psi}{\overset{\phi}{\rightleftarrows}} U_2$. U_1 in U_2 sta odprti in gosti, zato ju lahko enolično razširimo do racionalnih preslikav $X_1 \dashrightarrow X_2$ in $X_2 \dashrightarrow X_1$. Ti dve sta drugа drugi inverzni, ker sta ϕ in ψ druga drugi inverzni. ▨

Definicija: Naj bo $\phi: X \dashrightarrow Y$ dominantna racionalna preslikava med nerazcepnima kvaziprojektivnima raznoterostima in naj bo $g \in \mathbb{k}(Y)$. Ker je ϕ dominantna, lahko definiramo $g \circ \phi$, kar je racionalna funkcija na X. Pravimo ji povlek funkcije g s predpisom ϕ. Oznaka $\phi^*(g)$. Imamo torej preslikavo $\phi^*: \mathbb{k}(Y) \longrightarrow \mathbb{k}(X)$.

Naslednji rezultati imajo podobne dokaze kot smo jih naredili v primeru polinomskih preslikav.

Lema: ϕ^* je homomorfizem $\mathbb{k}$-algeber. Ker je $\mathbb{k}(Y)$ polje, je ϕ^* vedno injektiven (razen v trivialnih primerih, ko je ϕ^* ničeln).

Lema: Če sta $\phi: X \dashrightarrow Y$ in $\psi: Y \dashrightarrow Z$ dominantni racionalni preslikavi med nerazcepnima raznoterostima, potem je $\psi \circ \phi$ dominantna in velja $(\psi \circ \phi)^* = \phi^* \circ \psi^*$.

Izrek: Naj bosta X in Y nerazcepni afini raznoterosti. Potem preslikava $\phi \longmapsto \phi^*$ podaja bijektivno korespondenco med dominantnimi racionalnimi preslikavami $X \dashrightarrow Y$ in injektivnimi homomorfizmi algeber $\mathbb{k}(Y) \longrightarrow \mathbb{k}(X)$.

Posledica: Racionalna preslikava $X \dashrightarrow Y$ med nerazcepnima afinima raznoterostima je biracionalna ekvivalenca $\Leftrightarrow$ $\mathbb{k}(Y) \xrightarrow{\phi^*} \mathbb{k}(X)$ je izomorfizem algeber/polj.

Posledica: Racionalna preslikava $X \overset{\phi}{\dashrightarrow} Y$ med nerazcepnima projektivnima raznoterostima je biracionalna ekvivalenca $\Leftrightarrow$ $\mathbb{k}(Y) \xrightarrow{\phi^*} \mathbb{k}(X)$ je izomorfizem polj.

Opomba: Velja tudi za kvaziprojektivne raznoterosti.

Dokaz: Naj bo $X \subseteq \mathbb{P}^n$, $Y \subseteq \mathbb{P}^m$, $U_0 = D(x_0) \subseteq \mathbb{P}^n$, $U_0' = D(x_0) \subseteq \mathbb{P}^m$.
Praslika $\phi^{-1}(Y \cap U_0')$ je odprta in gosta v X, torej lahko predpostavimo, da je njen presek z U_0 neprazen in torej gost v $X \cap U_0$.
Zožitev $\phi|_{X \cap U_0}: X \cap U_0 \dashrightarrow Y \cap U_0$ je racionalna preslikava med nerazcepnima afinima raznoterostima. Po prejšnji posledica je ta zožitev biracionalna ekvivalenca $\Leftrightarrow$ $\mathbb{k}(Y \cap U_0') \longrightarrow \mathbb{k}(X \cap U_0)$ izomorfizem polj. Od zadnjič vemo, da je $\mathbb{k}(Y) = \mathbb{k}(Y \cap U_0')$ in $\mathbb{k}(X) = \mathbb{k}(X \cap U_0)$.
$X \cap U_0 \dashrightarrow Y \cap U_0'$ je biracionalna ekvivalenca $\Leftrightarrow$ $\mathbb{k}(Y) \longrightarrow \mathbb{k}(X)$ je izomorfizem polj. (*)
$X \cap U_0 \dashrightarrow Y \cap U_0'$ je biracionalna ekvivalenca $\Leftrightarrow$ obstajata odprti $V \subseteq X \cap U_0$ in $V' \subseteq Y \cap U_0'$, da je zožitev $\phi|_V : V \longrightarrow V'$ izomorfizem
$\Rightarrow X \dashrightarrow Y$ je biracionalna ekvivalenca.

Obratno je trdi res: če je $X \dashrightarrow Y$ biracionalna ekvivalenca, obstajata odprti $W \subseteq X$, $W' \subseteq Y$, ki sta izomorfni.
BŠS: $W \subseteq U_0$, $W' \subseteq U_0'$ $\Rightarrow$ $X \dashrightarrow Y$ je biracionalna ekvivalenca $\Leftrightarrow$ $X \cap U_0 \dashrightarrow Y \cap U_0'$ je biracionalna ekvivalenca $\Leftrightarrow$ (*). $\square$