5. L^p-PROSTORI

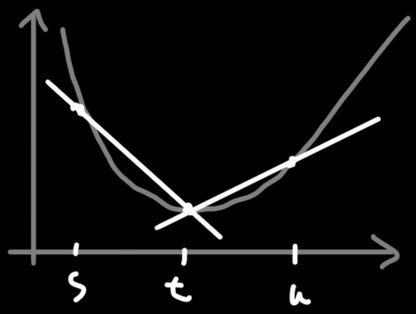
5.1. Konveksne funkcije in neenakosti

$\varphi : [a,b] \longrightarrow \mathbb{R}$ je konveksna, če za vse $x, y \in (a,b)$ in $\lambda \in [0,1]$ velja $\varphi((1-\lambda)x + \lambda y) \leq (1-\lambda)\varphi(x) + \lambda\varphi(y)$.
Pri nas je $-\infty \leq a < b \leq \infty$.

Pišemo $x = s$, $y = u$ in $t = (1-\lambda)s + tu$ in dobimo

$$\frac{\varphi(t) - \varphi(s)}{t-s} \leq \frac{\varphi(u) - \varphi(t)}{u-t}. \qquad (*)$$

„Nakloni sekant rastejo."



Iz $(*)$ se da dokazati, da je φ zvezna na odprtem intervalu, kar je pri nas (a,b).

Jensenova neenakost: Naj bo $(X, \mathcal{A}, \mu)$ verjetnostni prostor in $\varphi : (a,b) \longrightarrow \mathbb{R}$ konveksna funkcija. Naj bo $f \in L^1(\mu)$ taka funkcija, da je njena zaloga vrednosti vsebovana v (a,b).
Tedaj velja
$$\varphi\left(\int_X f\,d\mu\right) \leq \int_X (\varphi \circ f)\,d\mu.$$

Verjetnostna formulacija: $(\Omega, \mathcal{F}, \mathbb{P})$, φ konveksna funkcija, ... predpostavke $X \leftrightarrow f$
$$\varphi(\mathbb{E}X) \leq \mathbb{E}(\varphi(X))$$

Dokaz: Pišimo $t = \int_X f\,d\mu$

$a < t < b$

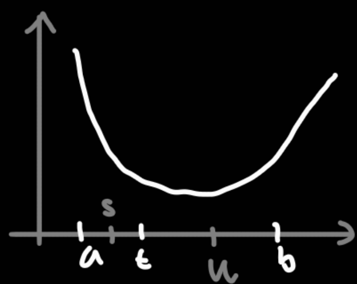
$$b > f(x) > a \quad \Big| \int_X$$
$$b = \int_X b\,d\mu \geq \int_X f(x)\,d\mu \geq \int_X a\,d\mu = a$$

$$\Rightarrow a \leq t \leq b$$

Ali je $t = b$ oziroma $\int_X f\,d\mu = \int_X b\,d\mu$ oziroma $\int_X (b-f)\,d\mu = 0$
$\Leftrightarrow b = f$ s.p., saj $b > f(x)$ $\forall x$

To ni možno. Podobno $a < t$.

Definiramo $\beta := \sup \frac{\varphi(t)-\varphi(a)}{t-a}$



Ker za vse $s \in (a,t)$ in vse $u \in (t,b)$ velja

$$\frac{\varphi(t)-\varphi(s)}{t-s} \leq \beta \leq \frac{\varphi(u)-\varphi(t)}{u-t}$$

Dobimo $\varphi(s) \geq \varphi(t) + \beta(s-t)$ za vse $s \in (a,b)$.

$s = f(x)$:

$$\varphi(f(x)) \geq \varphi(t) + \beta(f(x)-t)$$

$$\int_X (\varphi\circ f)\,d\mu \geq \underbrace{\int_X \varphi(t)\,d\mu}_{\varphi(t) = \varphi(\int_X f\,d\mu)} + \beta \underbrace{\int_X (f-t)\,d\mu}_{0}$$

Naj bo $X = \{p_1,\dots,p_n\}$ opremljen s potenčno σ-algebro in verjetnostno mero. Označimo $\alpha_j = \mu(\{p_j\})$ in $x_j = f(p_j)$, kjer je f funkcija. Če je φ konveksna, potem dobimo

$$\varphi\left(\int_X f\,d\mu\right) \leq \int_X (\varphi\circ f)\,d\mu$$

$$\varphi\left(\sum_{j=1}^n f(p_j)\,\mu(\{p_j\})\right) \leq \sum_{j=1}^n (\varphi\circ f)(p_j)\,\mu(\{p_j\}).$$

$$\varphi\left(\sum_{j=1}^n \alpha_j x_j\right) \leq \sum_{j=1}^n \alpha_j \varphi(x_j); \qquad \sum_{j=1}^n \alpha_j = 1, \quad 0 \leq \alpha_i \leq 1$$

Naj bo $f(x) = e^x$ in $y_j = e^{x_j}$

$$y_1^{\alpha_1} \cdots y_n^{\alpha_n} \leq \alpha_1 y_1 + \dots + \alpha_n y_n \qquad (**)$$

Vzemimo $\alpha_1 = \dots = \alpha_n = \frac{1}{n} \Rightarrow \sqrt[n]{y_1 \cdots y_n} \leq \frac{y_1 + \dots + y_n}{n}$ AG-neenakost

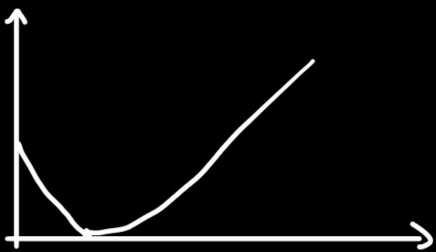
$n = 2$, $\alpha_1 = p$, $\alpha_2 = q$ v $(**)$ in dobimo $\frac{1}{p} + \frac{1}{q} = 1$ (p, q konjugirana eksponenta)

$y_1 = x^p$, $y_2 = y^q$ dobimo $xy \le \frac{1}{p}x^p + \frac{1}{q}y^q$ Youngova neenakost

Youngovo neenakost lahko dokažemo tudi z ANA1:

Za fiksen $y \ge 0$ opazuješ $f(x) = \frac{1}{p}x^p + \frac{1}{q}x^q - xy$

$$f'(x) = x^{p-1} - y \Rightarrow x = y^{\frac{1}{p-1}}$$



$\Rightarrow$ globalni minimum za f je dosežen, ko je $x = y^{p-1}$

$$f(x) = f\left(y^{\frac{1}{p-1}}\right) = \frac{1}{p} y^{\frac{p}{p-1}} + \frac{1}{q} y^q - y^{\frac{1}{p-1}+1}$$

$$= \frac{1}{p} y^{\frac{p}{p-1}} - y^{\frac{p}{p-1}} + \frac{1}{q} y^q$$

$$= \left(\frac{1}{p} - 1\right) y^{\frac{p}{p-1}} + \frac{1}{q} y^q$$

$$= -\frac{1}{q} y^{\frac{p}{p-1}} + \frac{1}{q}{}^{q} = 0$$

$$\Rightarrow \frac{p-1}{p} = 1 - \frac{1}{p} = \frac{1}{q}$$

Enakost v Youngovi neenakosti je dosežena $\Leftrightarrow$

$$x = y^{\frac{1}{p-1}} \quad \text{oziroma} \quad x^p = y^{\frac{p}{p-1}} = y^p$$

5.2. L^p-prostori

23. december 2025

Naj bo $(X, \mathcal{M}, \mu)$ merljiv prostor s pozitivno mero. Za $p \in [1, \infty)$ definirampo $L^p(\mu)$ kot množico vseh merljivih funkcij $f: X \longrightarrow \mathbb{C}$, da je $|f|^p \in L^1(\mu)$.

$|f|^p \in L^1(\mu) \Leftrightarrow \int_X |f|^p \, d\mu < \infty$.

Kasneje bomo pokazali, da je $\|f\|_p := \left(\int_X |f|^p \, d\mu\right)^{1/p}$ polnorma na $\mathcal{L}^p(\mu)$.

$\mathcal{L}^p(\mu)$ je vektorski prostor

i) $f, g \in \mathcal{L}^p(\mu) \Rightarrow f + g \in \mathcal{L}^p(\mu)$ ii) $f \in L^p(\mu)$, $\lambda \in \mathbb{C} \Rightarrow \lambda f \in L^p(\mu)$

Primer $p=1$ je znan, zato BSŠ $1<p<\infty$.

i) $t \longmapsto t^p$ je konveksna na $[0,\infty]$

$t_1, t_2 \geq 0$: $f\left(\frac{t_1+t_2}{2}\right) \leq \frac{1}{2} f(t_1) + \frac{1}{2} f(t_2)$

$$\left(\frac{t_1+t_2}{2}\right)^p \leq \frac{1}{2} t_1^p + \frac{1}{2} t_2^p$$

$F_1 = |f(x)|$, $g_1 = |g(x)|$

$$(|f(x)| + |g(x)|)^p \leq 2^{p-1}(|f(x)|^p + |g(x)|^p)$$

$$\int_X |f+g|^p d\mu \leq 2^{p-1}\left(\int_X |f|^p d\mu + \int_X |g|^p d\mu\right) < \infty$$

$$\|f+g\|_p^p \leq 2^{p-1}(\|f\|_p^p + \|g\|_p^p)$$

ii) $\underbrace{\int_X |\lambda f|^p d\mu}_{\|\lambda f\|_p^p} = |\lambda|^p \int_X |f|^p d\mu = |\lambda|^p \|f\|_p^p$

$$\Rightarrow \|\lambda f\|_p = |\lambda| \cdot \|f\|_p$$

$p \in (1,\infty) \Rightarrow$ če q zadošča $\frac{1}{p} + \frac{1}{q} = 1$, q imenujemo **konjugirani eksponent k p** oziroma p in q sta **konjugirana eksponenta**.

$p = q = 2$: $\frac{1}{2} + \frac{1}{2} = 1 \rightsquigarrow L^2(\mu)$

Hölderjeva neenakost: Naj bosta $p, q \in (1,\infty)$ konjugirana eksponenta. Naj bo $f \in L^p(\mu)$ in $g \in L^q(\mu)$. Tedaj je $fg \in L^1(\mu)$ in velja:

$$\left|\int_X fg\, d\mu\right| \leq \left(\int_X |f|^p d\mu\right)^{1/p} \cdot \left(\int_X |g|^q d\mu\right)^{1/q}$$

oziroma

$$\left|\int_X fg\, d\mu\right| \leq \|f\|_p \cdot \|g\|_q.$$

Posledica: Za $f \in L^p(\mu)$ in $g \in L^q(\mu)$, kjer sta $1<p,q<\infty$ konjugirana eksponenta, velja $\|fg\|_1 \leq \|f\|_p \|g\|_q$.

Dokaz Hölderjeve neenakosti: Najprej predpostavimo, da $f, g \geq 0$.
Vpeljimo $A := \left(\int_X f^p d\mu\right)^{1/p}$, $B := \left(\int_X g^q d\mu\right)^{1/q}$.

$\Rightarrow 0 \leq A, B < \infty$

Naša neenakost: $0 \leq \int_X fg\,d\mu \leq A \cdot B$

$A = 0 \Rightarrow f^p \equiv 0$ s.p. $\Rightarrow f \equiv 0$ s.p. $\Rightarrow fg \equiv 0$ s.p. $\Rightarrow 0 \leq 0 \leq 0B$ ✓

Če $B = 0$, podobno.

Zato BŠS $A, B > 0$. Definiramo $F := \frac{f}{A}$ in $G := \frac{g}{B}$.

$\Rightarrow F \in L^p(\mu)$, $\|F\|_p = 1$, $G \in L^q(\mu)$, $\|G\|_q = 1$

Po Youngovi neenakosti je

$$0 \leq FG \leq \frac{F^p}{p} + \frac{G^q}{q} \quad / \int$$

$$0 \leq \int_X FG \leq \int_X \left(\frac{F^p}{p} + \frac{G^q}{q}\right) d\mu = \frac{1}{p}\underbrace{\|F\|_p^p}_{1} + \frac{1}{q}\underbrace{\|G\|_q^q}_{1} = 1$$

$\int_X \frac{f}{A}\frac{g}{B}\,d\mu \leq 1$ oziroma $\int_X fg\,d\mu \leq AB \leq \|f\|_p^p \cdot \|g\|_q^q$

V splošnem:

$$\left|\int_X fg\,d\mu\right| \overset{\text{če je } fg \in L^1}{\leq} \int_X |fg|\,d\mu = \int_X \underset{L^q(\mu)}{\overset{L^p(\mu)}{|f| \cdot |g|}}\,d\mu \leq \| |f| \|_p \cdot \| |g| \|_q$$

$fg \in L^1$

$$= \|f\|_p \cdot \|g\|_q$$

$\blacksquare$

Lema: Naj bosta $p, q \in (1, \infty)$ konjugirana eksponenta. Tedaj za $f \in L^p(\mu)$ velja $\|f\|_p = \sup\{|\int_X fg\,d\mu| \mid \|g\|_q = 1\}$.

Opomba: $f \in L^p(\mu) \Rightarrow \varphi_f(g) := \int_X fg\,d\mu$; $g \in L^q(\mu)$ in p, q konjugirana eksponenta. Ta φ_f je omejen linearen funkcional na $L^q(\mu)$, saj je $|\varphi_f(g)| \leq \|f\|_p \|g\|_q$ (Hölder) $\Rightarrow \|\varphi_f\| \leq \|f\|_p$.

Po lemi je $\|\varphi_f\| = \|f\|_p$.

Dokaz: Označimo $s := \sup\{|\int_X fg\,d\mu| \mid \|g\|_q = 1\}$. Po Hölderjevi neenakosti je $\|f\|_p \geq s$. Definirajmo

$$g(x) = \begin{cases} \frac{|f(x)|^{p-1}}{\|f\|_p^{p-1}} \cdot \frac{\overline{f(x)}}{|f(x)|} & ; f(x) \neq 0 \\ 0 & ; f(x) = 0 \end{cases}$$

Funkcija g je dobro definirana, če $f \not\equiv 0$ s.p. Če je $f \equiv 0$ s.p., potem katerikoli g z $\|g\|_p = 1$ deluje.

BŠS $f \neq 0$ s.p.

$$\|g\|_q^q = \int_X |g|^q \, d\mu = \int_{\{x \mid f(x) \neq 0\}} \frac{|f|^{(p-1)q}}{\|f\|_p^{(p-1)q}} \, d\mu$$

$(p-1)q = pq - q = p$

$$= \int_{\{x \mid f(x) \neq 0\}} \frac{|f|^p}{\|f\|_p^p} \, d\mu = \frac{1}{\|f\|_p^p} \int_X |f|^p \, d\mu = 1$$

$$\int_X fg \, d\mu = \int_{\{x \mid f(x) \neq 0\}} f(x) \frac{|f(x)|^{p-1}}{\|f\|_p^{p-1}} \cdot \frac{\overline{f(x)}}{|f(x)|} \, d\mu = \frac{1}{\|f\|_p^{p-1}} \int_X |f(x)|^p \, d\mu$$

$$= \frac{1}{\|f\|_p^{p-1}} \cdot \|f\|_p^p = \|f\|_p. \qquad \blacksquare$$

<u>Neenakost Minkovskega</u>: Naj bo $1 \leq p < \infty$. Tedaj za $f, g \in L^p(\mu)$ velja $\|f+g\|_p \leq \|f\|_p + \|g\|_p$ oziroma

$$\left(\int_X |f+g|^p \, d\mu \right)^{1/p} \leq \left(\int_X |f|^p \, d\mu \right)^{1/p} + \left(\int_X |g|^p \, d\mu \right)^{1/p}.$$

<u>Dokaz</u>: Naj bosta $f, g \in L^p(\mu)$. Naj bo $h \in L^q(\mu)$ tak, da je $\|h\|_q = 1$.

$$\left| \int_X (f+g) h \, d\mu \right| \leq \int_X |fh + gh| \, d\mu \leq \int_X |fh| \, d\mu + \int_X |gh| \, d\mu$$

Ker je $\left| \int_X fh \, d\mu \right| \leq \int_X |fh| \, d\mu \leq \|f\|_p$, je

$$\left| \int_X (f+g) \, d\mu \right| \leq \|f\|_p + \|g\|_q.$$

Po lemi, ko naredimo supremum po vseh $h \in L^2(\mu)$ z $\|h\|_q = 1$, dobimo $\|f+g\|_p \leq \|f\|_p + \|g\|_p$. $\blacksquare$

<u>Opomba</u>: Alternativni dokaz neenakosti Minkovsega (ideja): $f, g \geq 0$

$$\int_X (f+g)^p \, d\mu = \int_X (f+g)(f+g)^{p-1} \, d\mu = \int_X \underset{\in L^p}{f} \, \underset{\in L^q}{(f+g)^{p-1}} \, d\mu + \int \underset{\in L^p}{g} \, \underset{\in L^q}{(f+g)^{p-1}} \, d\mu$$

2x uporabimo Hölderja in prekladamo norme in $\frac{1}{p} + \frac{1}{q} = 1$. $\blacksquare$

$p=\infty$, $L^\infty = ?$

merljiva $f \in L^\infty(\mu) :\Leftrightarrow \exists M \geq 0.\ |f(x)| \leq M$ za skoraj vsak $x \in X$.

Elementi $L^\infty(\mu)$ so t.i. bistveno omejene funkcije.

Za $f \in L^\infty(\mu)$ definiramo

$$\|f\|_\infty := \inf\{M \geq 0 \mid |f(x)| \leq M \text{ za skoraj vsak } x \in X\}$$

$\|f\|_\infty$ se imenuje bistveni supremum funkcije f in ga včasih označimo $\|f\|_\infty := \operatorname{ess\,sup}_{x \in X} |f(x)|$.

Lema: Naj bo $f \in L^\infty(\mu)$. Tedaj je $|f(x)| \leq \|f\|_\infty$ za skoraj vsak $x \in X$.

Dokaz: $A_n := \{x \in X \mid |f(x)| \geq \|f\|_\infty + \frac{1}{n}\}$. Po definiciji je $\mu(A_n) = 0$ $\forall n \in \mathbb{N}$. $\{x \in X \mid f(x) \leq \|f\|_\infty\} = \left(\bigcup_{n=1}^\infty A_n\right)^c$

Ker $\mu\left(\bigcup_{n=1}^\infty A_n\right) = 0$, je $|f(x)| \leq \|f\|_\infty$ s.p. $\blacksquare$

Dopolnjena Hölderjeva neenakost: Naj bo $p \in [1, \infty]$ in naj bo q konjugirani eksponent k p. Potem za $f \in L^p(\mu)$ in $g \in L^q(\mu)$ velja $\left|\int_X fg\, d\mu\right| \leq \|f\|_p \cdot \|g\|_q$.

Dokaz: i) $p \in (1, \infty) \rightsquigarrow$ Hölder

ii) $p = 1 \Rightarrow 1 = \frac{1}{p} + \frac{1}{q} \Rightarrow q = \infty$, $p = \infty \Rightarrow q = 1$

$$\left|\int_X fg\, d\mu\right| \leq \int_X |fg|\, d\mu = \int_X |f|\, \|g\|_\infty\, d\mu = \|f\|_1 \cdot \|g\|_\infty \quad \blacksquare$$

Lema: Naj bo $(f_n)_{n \in \mathbb{N}}$ v $L^\infty(\mu)$.

i) Če je $(f_n)_{n \in \mathbb{N}}$ Cauchyjevo v $L^\infty(\mu)$, potem $\exists A \in \mathcal{A}$, da je $\mu(A^c) = 0$ in $(f_n)_{n \in \mathbb{N}}$ enakomerno Cauchyjevo na A.

ii) Če $f_n \longrightarrow f$ v $L^\infty(\mu)$, potem $\exists A \in \mathcal{A}$, da je $\mu(A^c) = 0$ in $f_n \longrightarrow f$ enakomerno na A.

Cauchyjev pogoj: $\forall \varepsilon > 0.\ \exists N_\varepsilon.\ \forall m, n \geq N_\varepsilon.\ \|f_n - f_m\|_\infty < \infty$.

<u>Dokaz</u>: i) Za $n, m \in \mathbb{N}$ definiramo $A_{n,m} = \{x \in A \mid |f_n(x) - f_m(x)| > \|f_n - f_m\|_\infty\}$.
Po definiciji bistvene norme in po lemi, je $\mu(A_{n,m}) = 0$.
Definiramo $B := \bigcup_{n,m=1}^{\infty} A_{n,m}$. Tedaj $\mu(B) = 0$, ko $A := B^c$, pa velja $|f_n(x) - f_m(x)| \leq \|f_n - f_m\|_\infty$ povsod.
ii) Podobno kot i). ▨

Naj bo $p \in [1, \infty]$. V $L^p(\mu)$ vpeljemo relacijo $\sim$:

$$f \sim g :\Leftrightarrow f = g \text{ s.p.}$$

To je ekvivalenčna relacija. $L^p(\mu)/_\sim := L^p(\mu)$ zbraba notacije

$[f] \in L^p(\mu) \Rightarrow \|[f]\|_p := \|f\|_p$.
Če $1 \leq p < \infty$ in $f \sim g \Rightarrow \underbrace{\int_X |f|^p d\mu}_{= \|f\|_p^p} = \underbrace{\int_X |g|^p d\mu}_{= \|g\|_p^p}$

$p = \infty$ in $f \sim g \Rightarrow f = g$ s.p.

$$g(x) \overset{\text{s.p.}}{=} f(x) \overset{\text{s.p.}}{\leq} \|f\|_\infty \Rightarrow \|g\|_\infty \leq \|f\|_\infty$$

Podobno $\|f\|_\infty \leq \|g\|_\infty$.
Definiramo $[f] + [g] := [f + g]$ in $\lambda[f] := [\lambda f]$.
$f \sim f'$ in $g \sim g'$, potem $f + g \sim f' + g'$ in $\lambda f \sim \lambda f'$.
Zato je $L^p(\mu)$ tudi vektorski prostor in $\|\cdot\|_p$ na $L^p(\mu)$ je norma za $p \in [1, \infty)$.
$p = \infty$: $[f], [g] \in L^\infty(\mu)$:

$$\|[f] + [g]\|_\infty \overset{?}{\leq} \|[f]\|_\infty + \|[g]\|_\infty$$
$$\Updownarrow$$
$$\|f + g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$$

$|f(x) + g(x)| \leq |f(x)| + |g(x)| \leq \|f\|_\infty + \|g\|_\infty$ s.p.
Po definicije bistvene norme je $\|f + g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$.

$\|\lambda f\|_\infty = |\lambda|\,\|f\|_\infty$

$|\lambda f(x)| = |\lambda|\,|f(x)| \leq |\lambda|\,\|f\|_\infty$ s.p. $\Rightarrow \|\lambda f\|_\infty \leq |\lambda|\,\|f\|_\infty$

Če $\lambda \neq 0$: $|f(x)| = |\frac{1}{\lambda}\lambda f(x)| = \frac{1}{|\lambda|}\cdot|\lambda f(x)| \leq \frac{1}{|\lambda|}\|\lambda f\|_\infty$ s.p.

$\Rightarrow \|f\|_\infty \leq \frac{1}{|\lambda|}\|\lambda f\|_\infty$ oziroma $|\lambda|\,\|f\|_\infty \leq \|\lambda f\|_\infty$.

Izrek: $L^p(\mu)$ je Banachov prostor za $1 \leq p \leq \infty$.

Dokaz: $p=1$ (znano), $1<p<\infty$ (podobno kot $p=1$).

$p=\infty$: Naj bo $([f_n])_{n\in\mathbb{N}}$ Cauchyjevo zaporedje v $L^\infty(\mu)$.

$\forall \varepsilon>0.\exists n_\varepsilon.\forall n,m\geq n_\varepsilon.\ \|[f_n]-[f_m]\|_\infty < \varepsilon$

$$\|[f_n]-[f_m]\|_\infty = \|[f_n-f_m]\|_\infty = \|f_n-f_m\|_\infty$$

$\exists A\in\mathcal{A}.\ \mu(A^c)=0$ in $(f_n|_A)_{n\in\mathbb{N}}$ enakomerno Cauchyjevo

Naj bo $f := \lim_{n\to\infty} f_n|_A$ ← enakomerna limita na A

Ker so $f_n|_A$ merljive, je f merljiva na A.

$$\tilde{f}(x) := \begin{cases} f(x) & ; x\in A \\ 0 & ; x\in A^c \end{cases}$$

je merljiva na X in velja $\|f_n-\tilde{f}\|_\infty \longrightarrow 0$.

Torej $[f_n] \longrightarrow [\tilde{f}]$. ▨

Opomba: Naj bo $(X,\mathcal{A},\mu)$ merljiv prostor in $s: X \longrightarrow \mathbb{C}$ merljiva stopničasta funkcija. Naj bo $s = \sum_{k=1}^n \alpha_k \chi_{E_k}$ kanonična oblika za s. Tedaj je $|s| = \sum_{k=1}^n |\alpha_k|\chi_{E_k}$ in zato

$$\int_X |s|^p\,d\mu = \int_X \Big(\sum_{k=1}^n |\alpha_k|^p \chi_{E_k}\Big)d\mu = \sum_{k=1}^n |\alpha_k|^p \mu(E_k)$$

$s\in L^p(\mu) \Leftrightarrow \{x \mid s(x)\neq 0\}$ ima končno mero.

Posledica: Za $1\leq p<\infty$ je množica vseh stolpničastih funkcij iz $L^p(\mu)$ gosto v $L^p(\mu)$.

Dokaz: Najprej naj bo $F \geq 0$ v $L^p(\mu)$. Tedaj obstaja zaporedje $(s_n)_{n\in\mathbb{N}}$ stopničastih funkcij, da $0 \leq s_n \nearrow f$ po točkah.

$$\|f - s_n\|_p^p = \int_X |f - s_n|^p d\mu \xrightarrow{\text{glej spodaj}} 0$$

$$0 \leq s_n \nearrow f \Rightarrow f - s_n \searrow 0 \Leftrightarrow |f - s_n|^p \searrow 0, \ f^p \in L^1(\mu), \text{ LDK.}$$ ▨

Primer: $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)$; μ šteje točke

$L^\infty(\mu) = \{f : \mathbb{N} \longrightarrow \mathbb{C} \mid \|f\|_\infty < \infty\}$

$f \in L^\infty(\mu)$: $|f(x)| \leq \|f\|_\infty$ s.p.

Ker je $\mu(A) = 0 \Leftrightarrow A = \emptyset$, je f omejena.

$f \leftrightsquigarrow (f(n))_{n\in\mathbb{N}}$

$L^\infty(\mu) \leftrightsquigarrow \ell^\infty$ Banachov prostor z $\|(x_n)_{n\in\mathbb{N}}\| = \sup_{n\in\mathbb{N}} |x_n|$

$1 \leq p < \infty$: $L^p(\mu) = \{f : \mathbb{N} \to \mathbb{C} \mid \int |f|^p d\mu < \infty\}$.

$$\int_X |f|^p d\mu = \sum_{n=1}^\infty |f(n)|^p$$

$f \leftrightsquigarrow (f(n))_{n\in\mathbb{N}}$

$L^p \leftrightsquigarrow \ell^p$ Banachov prostor $\|(x_n)_{n\in\mathbb{N}}\| = \left(\sum_{n=1}^\infty |x_n|^p\right)^{1/p}$.

Kaj je $L^\infty(\mu)$?

Kaj je ℓ^∞? $\quad \ell^\infty = C_b(\underset{\substack{\uparrow \\ \mathbb{N} \text{ disk. top.}}}{\mathbb{N}}) \cong C(\overset{\substack{\text{Stone-Čech} \\ \downarrow}}{\beta}\mathbb{N})$

$L^\infty(\mu) \cong C(K)$ (brez dokaza)

5.3 Dualnost med L^p prostori

6. januar 2025

$\underset{\substack{\text{lep} \\ \text{lepa norma}}}{L^1}$, $\underset{\text{Hilbertov}}{L^2}$, $\underbrace{L^3, \ldots, L^p, \ L^q,}_{\substack{\text{zelo lepi} \\ \text{refleksivni}}}$ $\underset{\text{grd}}{L^\infty}$

X normiran prostor. Vprašanje: Ali na X obstaja zvezen linearni funkcional?

Da: po Hahn-Banachovem izreku obstaja veliko zveznih linearnih funkcionalov.

Definicija: Naj bo X normiran prostor in $f: X \longrightarrow \mathbb{F}$ ($\mathbb{R}$ ali $\mathbb{C}$) linearni funkcional. f je omejen, če $\exists M \geq 0$, da je
$$|f(x)| \leq M \cdot \|x\|.$$

Trditev: Naj bo $f: X \longrightarrow \mathbb{F}$ linearni funkcional. NTSE

i) f je omejen
ii) f je zvezen
iii) f je enakomerno zvezen
iv) f je Lipshitzev
v) f je zvezen v 0
vi) f je zvezen v neki točki

„Najmanjši" možni M, ki ga lahko vstavimo v neenakost imenujemo norma funkcionala f in jo označimo z $\|f\|$.

$$\begin{aligned}\|f\| &= \inf\{M \geq 0 \mid |f(x)| \leq M\|x\| \ \forall x \in X\} \\ &= \sup_{\|x\| \neq 0} \frac{|f(x)|}{\|x\|} \\ &= \sup_{\|x\|=1} |f(x)| \\ &= \sup_{\|x\| \leq 1} |f(x)| \\ &= \sup_{\|x\| < 1} |f(x)|\end{aligned}$$

operatorska norma funkcionala

Trditev: Če je X normiran, potem je $X^* := \{\text{omejeni linearni funkcionali na } X\}$ Banachov prostor glede na operatorsko normo.

Izrek: Za $\forall x \in X$. $\exists f \in X^*$, $\|f\| = 1$, da je $f(x) = \|x\|$.

Posledica: X^* loči točke na X.

Dokaz: $x, x' \in X$ in $x \neq x'$

$$\exists f \in X^*, \ \|f\| = 1 \qquad \underbrace{f(x - x')}_{= f(x) - f(x')} = \|x - x'\| \neq 0$$

$(L^3)^* \cong L^{3/2} \qquad \frac{1}{3}+\frac{1}{x}=1 \Rightarrow x=3/2$

$(L^p)^* \cong L^q \qquad \frac{1}{p}+\frac{1}{q}=1$

$f\in L^p(\mu),\ g\in L^q(\mu)$; $1\leq p\leq q\leq\infty$ in $\frac{1}{p}+\frac{1}{q}=1$

Hölder: $\left|\int_X fg\,d\mu\right| \leq \|f\|_p\cdot\|g\|_q$

$g\in L^q(\mu) \rightsquigarrow \varphi_g(f)=\int_X fg\,d\mu, \qquad \varphi_g : L^p(\mu)\longrightarrow \mathbb{F}$

$$|\varphi_g(f)|\leq \|f\|_p\cdot\|g\|_q \Rightarrow \varphi_g\in L^p(\mu)^* \text{ in } \|\varphi_g\|\leq\|g\|_q.$$

Trditev: Za $g\in L^q(\mu)$ je $\varphi_g\in L^p(\mu)^*$ in $\|\varphi_g\|\leq\|g\|_q$.

Trditev: Naj bosta p in q konjugirana eksponenta. Tedaj za $g\in L^q(\mu)$ velja naslednje:

i) Če je $q\in[1,\infty)$, potem je $\|\varphi_g\|=\|g\|_q$.

ii) Če $q=\infty$ in je μ σ-končna, potem je $\|\varphi_g\|=\|g\|_\infty$.

Opomba: Pod predpostavkami trditve je $L^q(\mu)\xrightarrow{\text{lin. izom.}} L^p(\mu)^*$.

Dokaz: i) Če $q\in(1,\infty)$ potem vemo po posledici Hölderjeve neenakosti, da je $\|\varphi_g\|=\|g\|_q$.

$$\|\varphi_g\|=\sup\left\{\left|\int_X fg\,d\mu\right| \;\middle|\; \|f\|_p=1\right\}=\|g\|_q.$$

$\underline{q=1}$: $g=0 \Rightarrow \varphi_g=0$ ✓

$g\neq 0$ s. p. $\Rightarrow f(x)=\begin{cases}\frac{\overline{g(x)}}{|g(x)|}; & g(x)\neq 0\\ 0; & \text{sicer}\end{cases}$

$\Rightarrow f\in L^\infty(\mu)$ in $\|f\|_\infty=1$ (bistvena norma)

$$\varphi_g(f)=\int_X fg\,d\mu=\int_X|g|\,d\mu=\|g\|_1.$$

ii) $\|\varphi_g\|=\|g\|_\infty$

- Če je $g=0$ s. p. $\Rightarrow \varphi_g=0$ in $[g]=0$
- BŠS $g\neq 0$ s. p., $\varepsilon>0$

$$F:=\{x\in X \mid |g(x)|\geq\|g\|_\infty-\varepsilon\}$$

Po definiciji bistvenega supremuma je $\mu(F)>0$.

Naj bo $(A_n)_{n\in\mathbb{N}}$ naraščujoče zaporedje merljivih množic s končno mero in $\bigcup_{n=1}^{\infty} A_n = X$.

$$F = \bigcup_{n=1}^{\infty} (A_n \cap F) \Rightarrow \mu(F) = \lim_{n\to\infty} \mu(A_n \cap F)$$

$\exists n \in \mathbb{N}.\ 0 < \mu(\underbrace{A_n \cap F}_{A}) < \infty$

Definiramo $f(x) := \begin{cases} \frac{1}{\mu(A)} \chi_A(x) \frac{\overline{g(x)}}{|g(x)|} & ; g(x) \neq 0 \\ 0 & ; \text{sicer} \end{cases}$.

$\|f\|_1 = \int_X |f| \, d\mu = \frac{1}{\mu(A)} \int_A \chi_A \, d\mu = 1$

$\|\varphi_g\| \geq \|\varphi_g(f)\| = |\int_X fg \, d\mu| = \frac{1}{\mu(A)} | \int_A |g| \, d\mu \geq \frac{1}{\mu(A)} (\|g\|_\infty - \varepsilon) \mu(A) = \|g\|_\infty - \varepsilon$

Pošljimo $\varepsilon \to 0$: $\|\varphi_g\| \geq \|g\|_\infty$. Obratno vedno velja. $\square$

Izrek: Naj bo μ σ-končna mera in $p \in [1, \infty)$. Tedaj za $\forall \varphi \in L^p(\mu)^*$ obstaja skoraj povsod enolično določena funkcija $g \in L^q(\mu)$, da je $\varphi = \varphi_g$. $\varphi(f) = \int_X fg \, d\mu$

Dokaz: Želimo poiskati funkcijo $g \in L^q(\mu)$, da je

$$\varphi(f) = \int_X fg \, d\mu.$$

Enoličnost: Če $\varphi_g = \varphi_{g'}$, potem je $\varphi_{g-g'} = 0$.

$$0 = \|\varphi_{g-g'}\| = \|g - g'\|_q \Leftrightarrow g - g' = 0 \quad \text{s.p.}$$

Eksistenca: Najprej predpostavimo, da je μ končna mera.
Za $A \subseteq X$ merljiva definiramo

$$\lambda(A) := \varphi(\chi_A). \quad (\text{ok definirana, saj } \mu(A) < \infty \Leftrightarrow \chi_A \in L^p)$$

λ je kompleksna mera

$$\lambda(\emptyset) = \varphi(\chi_\emptyset) = \varphi(0) = 0$$

$(A_n)_{n\in\mathbb{N}}$ paroma disjunktne v $\mathcal{A}$ in $A = \bigcup_{n=1}^{\infty} A_n$

$$\lambda(A) = \sum_{n=1}^{\infty} \lambda(A_n) \Leftrightarrow \varphi(\chi_A) = \sum_{n=1}^{\infty} \varphi(\chi_{A_n})$$

$\sum_{n=1}^{\infty} \varphi(\chi_{A_n}) = \varphi\left(\sum_{n=1}^{\infty} \chi_{A_n}\right)$?

Ker je φ zvezna, moramo preveriti, da vrsta $\sum_{n=1}^{\infty} \chi_{A_n}$ konvergira v $L^p(\mu)$.

$$\|\chi_A - \sum_{j=1}^{n} \chi_{A_j}\|_p^p = \sum_{j=n+1}^{\infty} \int_X \chi_{A_j}^p \, d\mu$$

$$= \sum_{j=n+1}^{\infty} \int_X \chi_{A_j} \, d\mu$$

$$= \sum_{j=n+1}^{\infty} \mu(A_j) \longrightarrow 0, \text{ saj } \infty < \mu(A) = \sum_{n=1}^{\infty} \mu(A_n)$$

Zato $\sum_{n=1}^{\infty} \chi_{A_n}$ konvergira in po zveznosti

$$\underset{\lambda\left(\bigcup_{n=1}^{\infty} A_n\right)}{\underset{\parallel}{\varphi\left(\sum_{n=1}^{\infty} \chi_{A_n}\right)}} = \sum_{n=1}^{\infty} \varphi(\chi_{A_n}) = \sum_{n=1}^{\infty} \lambda(A_n)$$

Torej je λ kompleksna mera.

$\lambda \ll \mu$

$\mu(A) = 0 \Rightarrow \chi_A = 0$ v $L^p(\mu) \Rightarrow \lambda(A) = \varphi(\chi_A) = \varphi(0) = 0$

Po Radon-Nikodymovem izreku $\exists g \in L^1(\mu)$, enolično določena s.p., da je

$$\lambda(A) = \int_A g \, d\mu \quad \forall A \in \mathcal{A}$$
$$\parallel \qquad\qquad \parallel$$
$$\varphi(\chi_A) = \int_A \chi_A g \, d\mu \quad \forall A \in \mathcal{A}$$

Torej $\varphi(f) = \int_X f g \, d\mu$ $\forall f$ karakteristična $(f = \chi_A)$

Zaradi linearnosti je $\varphi(s) = \int s g \, d\mu$ $\forall s \in L^p(\mu)$ stopničasta.

Če dokažemo, da je $g \in L^q(\mu)$, potem se zvezna funkcionala $f \longmapsto \varphi(f)$ in $f \longmapsto \int_X f g \, d\mu$ ujemata na stopničastih funkcijah iz $L^p(\mu)$. Ker so te funkcije goste v $L^p(\mu)$, se zaradi zveznosti funkciji ujemata povsod. Zato $\varphi(f) = \int_X f g \, d\mu$ $\forall f \in L^p(\mu)$.

$g \in L^q(\mu)$

$\forall n \in \mathbb{N}$ definiramo $X_n := \{x \in X \mid |g(x)| \leq n\}$

$\mu(X_n) < \infty \Rightarrow g_n := \chi_{X_n} \cdot g \in L^q(\mu)$, saj omejena in X_n ima končno mero

(Intermezzo: $A \subseteq X$ merljiva $\Rightarrow L^p(A) \overset{\text{izometrično}}{\hookrightarrow} L^p(X)$ na naraven način
$$f \in L^p(\mu) \longleftrightarrow \tilde{f}(x) = \begin{cases} f(x) ; & x \in A \\ 0 ; & \text{sicer} \end{cases}$$
)

Definirajmo φ_n na $L^p(\mu)$: $\varphi_n(f) = \int_X f g_n \, d\mu$ je omejen, saj $g_n \in L^q(\mu)$ in $\|\varphi_n\| = \|g_n\|_q = \| g|_{X_n} \|_q$.

Oglejmo si $A \subseteq X_n$ merljiva. $\varphi(\chi_A) = \int_A g \, d\mu = \varphi_n(\chi_A)$
$$\varphi(s) = \varphi_n(s) \quad \forall s \in L^p(X_n) \text{ stopničasta}$$
φ in φ_n sta omejena in se ujemata na stopničastih funkcijah v $L^p(X_n)$. Po omejenosti je $\varphi|_{L^p(X_n)} = \varphi_n$.
$$\| g|_{X_n} \|_q = \|\varphi_n\| = \| \varphi|_{L^p(X_n)} \| \leq \|\varphi\|$$

$\| g|_{X_n} \|_q \longrightarrow \|g\|_q$

$|g|\chi_n \nearrow |g| \quad \overset{\text{LMK}}{\Rightarrow} \quad \int_X |g|^q \chi_{A_n} d\mu \nearrow \int_X |g|^q d\mu$

$$\Rightarrow \int_X |g| d\mu \nearrow \|\varphi\|$$
$$\Rightarrow g \in L^q(\mu)$$

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Naj bo sedaj μ σ-končna. Tedaj $\exists (X_n)_{n \in \mathbb{N}}$ naraščajoče zaporedje množic s končno mero v $\mathcal{A}$, da je $X = \bigcup_{n=1}^{\infty} X_n$.
$$X_1 \subseteq X_2 \subseteq X_3 \subseteq \cdots \quad \longleftrightarrow \quad L^p(X_1) \subseteq L^p(X_2) \subseteq L^p(X_3) \subseteq \cdots$$

$A \subseteq B \longleftrightarrow L^p(A) \subseteq L^p(B)$ pri čemer $f \in L^p(A)$ razširimo z 0 na $B \setminus A$.
$\varphi \in (L^p(X))^* \rightsquigarrow \varphi_n := \varphi|_{L^p(X_n)}$ je omejen funkcional.
Velja $\|\varphi_n\| \leq \|\varphi\|$.

$\mu(X_n) < \infty \Rightarrow \exists g_n \in L^q(X_n)$; g_n enolično določena skoraj povsod, da je $\varphi_n(f) = \int_{X_n} f g_n \, d\mu \quad \forall f \in L^p(X_n)$.
Ker je $\varphi|_{L^p(X_n)} = \varphi_n \ \forall n \in \mathbb{N}$ in $\varphi_n|_{L^p(X_m)} = \varphi_m \ \forall n \geq m$, je $g_n|_{X_m} = g_m$ skoraj povsod na X_m.

Zato obstaja množica N z ničelno mero, da je $\forall n \geq m$
$$g_n|_{X_m} = g_m \quad \text{povsod na } A := N^c \qquad (*)$$

$$x \in X \Rightarrow g(x) = \begin{cases} g_n(x); & x \in X_n \cap A \\ 0; & x \in N \end{cases}$$

Zaradi $(*)$ je g dobro definiran.

Ker $|g_n| \nearrow |g|$, je $\underbrace{\int_X |g_n|^q d\mu}_{= \|\varphi_n\|^q \leq \|\varphi\|^q} \overset{LMK}{\nearrow} \int_X |g|^q d\mu \Rightarrow \|g\|_q \leq \|\varphi\|$

$\varphi(f) = \int_X fg\, d\mu \quad \forall f \in L^p(\mu)$

$\varphi_n(f) = \int_X \chi_{X_n} f g_n\, d\mu \quad \forall f \in L^p(X_n)$

$f \in L^p(X) \rightarrow f\chi_{X_n} \in L^p(X_n)$

$\varphi(f\chi_{X_n}) = \varphi_n(f\chi_{X_n}) = \int_X \chi_{X_n} f g_n\, d\mu$

$f\chi_{X_n} \longrightarrow f$ v $L^p(\mu)$

$\|f\chi_{X_n} - f\|_p^p = \int_X |f\chi_{X_n} - f|_p^p\, d\mu \xrightarrow{LDK} 0$

$|f\chi_{X_n} - f|^p \leq |f|^p \in L^1(\mu)$

φ omejen (zvezen) $\Rightarrow f\chi_{X_n} \longrightarrow f \Rightarrow \varphi(f\chi_{X_n}) \longrightarrow \varphi(f)$.

$\int_X \chi_{X_n} f g_n\, d\mu \longrightarrow \int_X fg\, d\mu$

$\chi_{X_n} f g_n \longrightarrow fg$ po točkah

$|\chi_{X_n} f g_n| \leq 1 f |g| \in L^1(\mu)$

Zato po LDK velja

Opomba: i) $1 \leq p < \infty$ in μ σ-končna

$(L^p(\mu))^* \cong L^q(\mu)$ izometrično izomorfen

$f \in (L^p(\mu))^* \Leftrightarrow \exists g \in L^q(\mu): \varphi(f) = \int_X fg\, d\mu$

ii) $1 < p < \infty$: potem i) vedno drži (brez σ-končnosti), torej

$(L^p(\mu))^* \cong L^q(\mu$

$Y \subseteq X$ σ-končna podmnožica

$\varphi|_{L^p(Y)} =: \varphi_Y \qquad \varphi_Y(f) = \int_Y f g_Y\, d\mu|_Y$

$\Rightarrow \|g_Y\| = \|\varphi_Y\| \leq \|\varphi\| \qquad g_Y$ z max. normo je iskani g

iii) $L^1(\mu) \hookrightarrow (L^\infty(\mu))^*$ izometrično, „redko surjektivno"

$X = \{0,1\}$: $\mu(\{0\}) = 1$, $\mu(\{1\}) = \infty$

$L^\infty(\mu)$ je 2-dim.

$f \in L^1(\mu) \Leftrightarrow \int_X |f| \, d\mu = |f(0)| \cdot 1 + |f(1)| \cdot \infty < \infty$

$f \in L^1(\mu) \Leftrightarrow f(1) = 0$

$\Rightarrow L^1(\mu) \cong \mathbb{F}$, $L^\infty(\mu) \cong \mathbb{F}^2$

iv) $L^\infty(\mu) \hookrightarrow (L^1(\mu))^*$ ni izometrično, če μ ni lepa

(μ ni semi-končna)