Operator theory

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PART I

Compact and Fredholm operators

Preliminaries

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Def: (X, ρ) - metric space if X-set, and ρ is a metric:

i) $\rho(x,y) \geq 0$ $\forall x,y \in X$. $\rho(x,y) = 0 \Leftrightarrow x=y$

ii) $\rho(x,y) = \rho(y,x)$ $\forall x,y \in X$

iii) $\rho(x,z) \leq \rho(x,y) + \rho(y,z)$ $\forall x,y,z \in X$

Def: $U \subseteq X$ is open if $\forall x \in U$. $\exists \delta > 0$. s.t. $B(x,\delta) \subset U$ $\quad (B(x,\delta) = \{y \in X \mid \rho(x,y) < \delta\})$

Def: $K \subset X$ is compact if every open cover $\{U_\alpha\}_{\alpha \in I}$ of K has a finite subcover.

cover: $\{U_\alpha\}_\alpha$ is a cover of K : if $\bigcup\limits_{\alpha \in I} U_\alpha \supset K$)

Def: A precompact set $A \subset X$ is a set $A \subset X$ s.t. $\underset{\substack{\text{closure} \\ \text{of } A \\ \text{in } X}}{\overline{A}}$ is compact.

__Def__: $\{x_j\}_{j \geq 1}$ is Cauchy sequence in X if
$\forall \varepsilon > 0.\ \exists N = N(\varepsilon).\ \rho(x_j, x_k) < \varepsilon\ \forall j, k \geq N(\varepsilon).$

__Def__: X is complete if $\forall$ Cauchy $\{x_j\}_{j \geq 1} \subset X$.
$\exists x \in X.\ \rho(x_j, x) \to 0$ as $j \to \infty$
(Every Cauchy sequence converges.)

__Ex.__ $\left(\mathbb{R}^n, \rho_{\mathbb{R}^n}(\{x_i\}, \{y_j\}) := \sqrt{\sum |x_i - y_i|}\right)$ – complete metric space

__Ex.__ $(\mathbb{Q}, \rho_{\mathbb{R}^n})$ – metric space but non-complete

__Ex.__ $[0, 1]$ is a compact-subset of $(\mathbb{R}, \rho_{\mathbb{R}})$

__Th__: $K \subseteq \mathbb{R}^n$ is compact $\Leftrightarrow$ closed and bounded

__Def__: $A \subset (X, \rho)$ is bounded if $\exists x \in X.\ \exists R > 0.\ A \subset B(x, R)$

__Ex.__: $(X, \rho) = \ell^2(\mathbb{Z}) = \left(\left\{\{\underset{\underset{\mathbb{C}}{\uparrow}}{x_j}\}_{j \in \mathbb{Z}} \mid \sum_{j \in \mathbb{Z}} |x_j|^2 < \infty\right\},\right.$

$$\rho(\{x_j\}, \{y_j\}) = \sqrt{\sum_{j \in \mathbb{Z}} |x_j - y_j|^2}$$

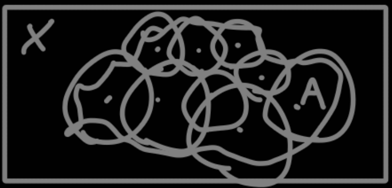
$$B[0, 1] = \{y \in \ell^2(\mathbb{Z}) : \rho(0\ x) \leq 1\}$$

– bounded, closed but not compact HW

__Theorem__: Let (X, ρ) be a complete metric space, $A \subset X$. The following assertions are equivalent:

i) A is precompact

ii) $\forall \varepsilon > 0.\ \exists$ a Finite ε-net $\{x_j\}_{j=1}^{N_\varepsilon}$ in A, that is, $\bigcup_{j=1}^{N_\varepsilon} B(x_j, \varepsilon) \supset A$.



iii) $\forall \{x_j\}_{j\geq 1} \subset A$ there is a converging subsequence to some element $x \in X$.

Proof: i) $\Rightarrow$ ii) $\{U_x\}_{x\in A} = \{B(x,\varepsilon)\}_{x\in A}$ - open cover of A.

- is an open cover of $\bar{A}$:

$(\forall y \in \bar{A}.\ \exists x \in A.\ \rho(x,y) < \varepsilon/2 \Rightarrow y \in B(x,\varepsilon))$

definition of closure

$\Rightarrow \exists \ \{U_{x_j}\}_{j=1}^{N}$ - finite subcover of $\bar{A}$

$\Rightarrow \ \{U_{x_j}\}_j^N$ is a ε-net in A

ii) $\Rightarrow$ iii) Observe that $\forall \varepsilon > 0$. any sequence $\{y_i\} \subset A$ has an infinite subsequence that is contained in some $B(x,\varepsilon)$. (we have a finite ε-net)

Assume that $\{y_k\}$ is arbitrary in A.

Cantor diagonal process

$y_1, y_2, y_3, y_4, \dots$

$y_1, y_{k_2}, y_{k_3}, \dots$ where $\{y_{k_j}\} \subset B(x_2, \frac{1}{2})$

$y_1, y_{k_2}, y_{kj_3}, y_{kj_4}$ where $\{y_{k_{j_s}} \subset B(\overset{\in X}{x_3}, 1/3)$

$y_1, y_{k_2}, y_{kj_3}, y_{kjs_4}$ $\dots$

Consider $z_1 = y_1$

$z_2 = y_{k_2}$

$z_3 = y_{kj_3}$

$z_4 = y_{kjs_4}$

$\vdots$

Claim: $\{z_j\}$ is a Cauchy sequence.

Indeed $\underset{k<j}{}$ $\rho(z_j, z_k) < \frac{1}{k}$

because $z_j, z_k \in B(x_k, \frac{1}{k})$ $\rho(z_j, z_k) < \frac{2}{K} \xrightarrow{} \infty$ $\Rightarrow$ ok

X is complete $\Rightarrow$ $\{z_j\}$ converges

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iii) $\Rightarrow$ i):

Plan: a) A contains a dense countable subset

a)+b)+c) under assumption 3)

b) If $\{V_\alpha\}_{\alpha \in I}$ is an open cover of $\bar{A}$ $\Rightarrow \exists \{V_{\alpha_j}\}_{j \in J}$ - an open coutable subcover of $\bar{A}$

c) $\Rightarrow$ $\{U_{\alpha_j}\}_{j=1}^{r}$ is a cover of $\bar{A}$

a) Observe that $\forall \varepsilon > 0$ there exists at most $N(\varepsilon)$ points $A \supset \{y_j(\varepsilon)\}_{1 \le j \le N(\varepsilon)}$ s.t. $\rho(y_k(\varepsilon), y_j(\varepsilon)) > \varepsilon$ $\forall k \ne j$.

(If this is not true, then $\exists \{y_j(\varepsilon)\}_{j=1}^{\infty}$ such that $\rho(y_j(\varepsilon), y_k(\varepsilon)) > \varepsilon$ and it cannot contain a convergent subsequence by Cauchy criterion.)

Now $E = \left\{ y_k\{\frac{1}{n}\} \mid 1 \le k \le N(\frac{1}{n}), n \ge 1 \right\}$ is a dense countable subset.

(E is dense since $\forall n . \forall x \in A . \min\limits_{1 \le k \le N(\frac{1}{n})} (\rho(x, y_k(\frac{1}{n})) \le \frac{1}{n}$ by construction)

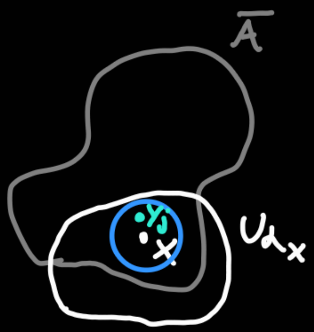
b) Assume that $\{U_\alpha\}_{\alpha \in I}$ is some open cover of $\bar{A}$.

For every $x \in \bar{A}$ define

$$\varepsilon(x) \overset{(*)}{:=} \frac{\sup\{\varepsilon > 0 \mid B(x, \varepsilon) \subset U_\alpha \text{ for some } \alpha\}}{2} > 0$$

Claim: if $\{y_j\}_{j=1}^{\infty}$ is a countable dense subset in A, then

$\{B(y_j, \varepsilon(y_j))\}_{j=1}^{\infty}$ is an open cover for $\bar{A}$.



Take $x \in \bar{A}$ $\exists \alpha_x . \ x \in U_{\alpha_x}$, and let $\eta(x) > 0$ such that $B(x, \eta(x)) \subset U_{\alpha_x}$ (U_{α_x} is open $\Rightarrow \eta(\varepsilon)\ \exists$)
Find y_j such that $\rho(x, y_j) < \frac{\eta(x)}{10}$.
Then, since $\eta(x) \leq 2\varepsilon(x)$, $x \in B(y_j, \varepsilon(y_j)) \Leftrightarrow \rho(x, y_j) < \varepsilon(y_j)$

$\varepsilon(y_j) \geq \frac{\eta(x)}{10}$ - Because $B(y_j, \frac{\eta(x)}{5}) \subset U_{\alpha_x}$ by triangle inequality and $\varepsilon(y_j)$ satisfies ⊛.

Since $\{B(y_j, \varepsilon(y_j))\}_{j=1}^{\infty}$ is an open cover for $\bar{A}$, then $\{U_{\alpha_{y_j}}\}_{j=1}^{\infty}$ is an open cover for $\bar{A}$, where $U_{\alpha_{y_j}}$ is the set U_α from the definition of $\varepsilon(y_j)$
(that is, $U_\alpha \underset{(**)}{\supseteq} B(y_j, \varepsilon(y_j))$).

Since we have (**), $\bigcup_{j=1}^{\infty} U_{y_{\alpha_i}} \supset \bar{A}$.

c) <u>Claim</u>: $\exists N . \{U_{\alpha_{y_j}}\}_{j=1}^{N}$ is a cover of $\bar{A}$.

Suppose this is not the case $\Rightarrow \forall j \geq 1 . \exists x_j \in A_j \setminus \bigcup_{k=1}^{j} U_{y_j}$.
Consider $\{x_j\}_{j=1}^{\infty}$, and assume that the sequence $\{x_{j_k}\}$ - convergent to some $x \in X$. Note that $x \in \bar{A}$ ($x_j \in \bar{A}$).
$\Rightarrow \exists j_* . \ x \in U_{\alpha_{j_*}} \Rightarrow \exists \delta > 0$. s.t. $B(x, \delta) \subset U_{\alpha_{j_*}}$, but $x_{j_k} \notin U_{\alpha_{j_*}}$ for large k by construction.
(in particular, $x_{j_k} \notin B(x, \delta)$, hence $\rho(x, x_{j_k}) > \delta$, but this contradicts the fact that $x_{j_k} \longrightarrow x$).
We have shown that $\{x_j\} \subset \bar{A}$ cannot have a convergent subsequence.
Then if $\tilde{x}_j \in A . \ \rho(\tilde{x}_j, x_j) < \frac{1}{j}$, then $\{\tilde{x}_j$ also has no convergent subsequence. So, we assumed there is no

finite subcover $\{U_{\alpha_{y_j}}\}$ and found a sequence $\{\tilde{x}_j\} \subset A$ that has no converging subsequence, a contradiction with 3). Therefore 3) $\Rightarrow$ 1). ▨

Examples of compact sets and their properties:

1) $K \subset (X, \rho)$ is compact $\Rightarrow$ K is bounded

Indeed, if K is not bounded, then $\{B(x, n)\}_{n \geq 1}$ is an open cover without a finite subcover.

2) $K \subset (X, \rho)$ is compact, then it is closed
($\Leftrightarrow$ $\{x_j\} \subset K$ such that $x_j \to x$ in (X, ρ) we also have $x \in K$)

Let's check that $X \setminus K$ is open. Take $y \in X \setminus K$, take $x \in K$, let $\delta(x) > 0$. $B(x, \delta(x)) \cap B(y, \delta(x)) \neq \emptyset$
$\{B(x, \delta(x))\}_{x \in K}$ is an open cover, let $\{B(x_j, \delta(x_j))\}_{j=1}^{N}$ be a finite subcover, then $\delta := \min\limits_{1 \leq j \leq K} \delta(x_j)$, $B(y, \delta) \cap K = \emptyset$
$\Rightarrow$ $X \setminus K$ is open.

Another proof: Suppose $\{y_j\} \subset K$ s.t. $y_j \to y$, $y \notin K$.

$$U_j = \{x \in X \mid \rho(x, y) > \frac{1}{j}\}$$

$\{U_j\}_{j=1}^{\infty}$ open cover, $\{U_{j_k}\}_{k=1}^{N}$ finite subcover

$\varepsilon := \min\limits_{1 \leq k \leq N} \left(\frac{1}{j_k}\right)$, $\rho(x, y) > \varepsilon$ $\forall x \in K$ contradiction ▨

linear space = vector space

3) Let X be a finite-dimensional complete linear normed space. Then $E \subset X$ is compact $\Leftrightarrow$ E is closed and bounded.

($\Rightarrow$): Examples 1+2.

($\Leftarrow$): $X = \{ \sum_1^N a_k e_k \mid c_k \in \mathbb{C} \}$, $N = \dim X$, $\{e_k\}_1^N$ - basis

$$\varphi\left(\sum a_k e_k\right) = \max_{1 \le k \le N} |c_k| \quad \text{- norm on } X$$

Since all norms on finite dimensional vec. spaces are equivalent.

$\exists A, B > 0.$ $A\|x\| \le \varphi(x) \le B\|x\|$ $\forall x \in X.$

in particular the set $\left\{ \{c_k(x)\}_1^N, \ \begin{array}{l} x \in E \\ x = \sum c_k(x) e_k \end{array} \right\}$

is bounded (= bdd) in $\mathbb{C}^n$ for every bounded $E \subset X$.

$$\sup_{x \in E} \varphi(x) \le B \cdot \sup_{x \in E} \|x\| < \infty$$

$\Rightarrow$ for any sequence $\left\{ \{c_k(x_j)\}_{k=1}^N \right\}_{j=1}^\infty$ one can extract a converging subsequence in $\mathbb{C}^n$, i.e.

$$c_k(x_{j_n}) \longrightarrow c_k \qquad n \to \infty.$$

But then $\sum_1^N c_k(x_{j_n}) e_k \longrightarrow \sum c_k e_k$ in X

$\Rightarrow$ bounded subsets are precompact in X

$\Rightarrow$ bdd + closed sets are compact

4) $\ell^2(\mathbb{Z}) = \left\{ \{c_k\}_{k \in \mathbb{Z}} \mid \sum |c_k|^2 < \infty \right\}$

$$\|\{c_k\}\| = \sqrt{\sum_{k=1}^\infty |c_k|^2}$$

$B[0,1] = \left\{ \{c_k\}_{k \in \mathbb{Z}} \in \ell^2(\mathbb{Z}) \mid \|\{c_k\}\| < 1 \right\}$

Then this set is bounded, closed, but neither compact nor precompact.

Proof: there is no finite $\frac{1}{2}$-net in $B[0,1]$, beca $\rho(e_k, e_j) > 1/2$ for $e_k = (0, \dots, 0, \underset{\substack{\wedge \\ \text{kth place}}}{1}, 0, \dots, 0)$.

Definition: X is a Banach space if it is a linear normed space such that X is complete with respect to this norm.

Example: Let (K, ρ) be a metric compact space.

$$C(K) := \{ f : K \to \mathbb{C} \mid \text{cont. in the metric } \rho \}$$

$$(\Leftrightarrow f(x_j) \to f(x) \quad \forall x_j \to x \text{ in } (K, \rho))$$

$$\|f\|_{C(K)} = \|f\| := \max_{x \in K} |f(x)| \quad \text{- norm in } C(K)$$

Theorem: [Arzela-Ascoli]: Assume that K is a complete compact metric space. $E \subset C(K)$ is precompact $\Leftrightarrow$

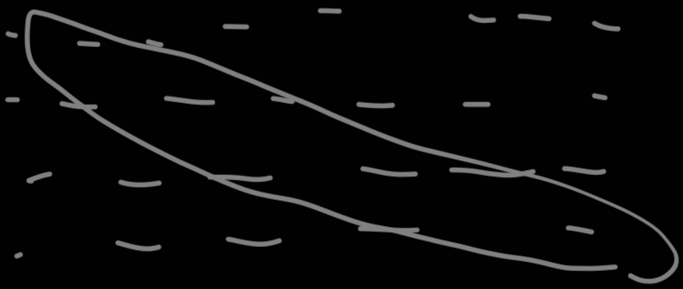
$$\Leftrightarrow \begin{cases} 1) & E \text{ is bounded in } C \\ 2) & \text{Functions in } E \text{ are equicontinuous, that is,} \\ & \forall \varepsilon > 0, \ \exists \delta_\varepsilon > 0. \ |f(x) - f(y)| < \varepsilon \quad \forall x, y \in K. \ \rho(x, y) < \delta_\varepsilon \\ & \qquad \forall f \in E \end{cases}$$

We will need $1) + 2) \Rightarrow$ precompactness.

Proof: Find a dense sequence $\{x_j\}$ in K.
(such sequence exists because K is compact)
Then take $\{f_n\}$ arbitrary sequence in E.
We want to find a converging subsequence of $\{f_n\}$
(then E - precompact)
For this find a subsequence $\{f_{n_k}\}$ such that

$$f_{n_k}(x_j) \longrightarrow f(x_j) \quad \text{for every } j$$

(Cantor diagonalization process + uniform boundedness)



look at the proof from the 1st lecture

Claim: f_{n_k} is Cauchy sequence in $C(K)$.

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Aim: $\|f_{n_s}-f_{n_m}\|_{C(K)} \longrightarrow 0$ as $s,m\to\infty$

For simplicity let $g_s = f_{n_s}$ $s\ge 1$.

Idea:

$$|g_s(x)-g_m(x)| \le \underset{\substack{\le \varepsilon/3 \\ \text{For all } s \text{ if } x_j \\ \text{is close to } x}}{|g_s(x)-g_s(x_j)|} + \underbrace{|g_s(x_j)-g_m(x_j)|}_{\substack{\text{take } s,m \text{ large} \\ \text{enough:} \\ \le \varepsilon/3}} + \underbrace{|g_n(x_j)-g_m(x)|}_{\substack{\le \varepsilon/3 \\ \text{For all } s \text{ if } x_j \\ \text{is close to } x \\ \text{(equicontinuity)}}}$$

To make the idea work we need to check that in this construction we can deal only with finite number of points x_j, $j=1,\dots,N(\varepsilon)$.

For this it suffices to find $N(\delta_\varepsilon)$ such that $\rho(x,x_j)<\delta_\varepsilon$ for every $x\in K$ and x_j, $j=1\dots N_\varepsilon$.
($\{x_j\}_{j=1}^{N(\delta_\varepsilon)}$ is δ_ε-net).

So, it remains to show that if $\{x_j\}_{j=1}^{\infty}$ is dense then $\forall\delta_\varepsilon>0.\ \exists N(\delta_\varepsilon).\ \{x_j\}_{j=1}^{N(\delta_\varepsilon)}$ is a δ_ε-net.
To this end, let $\{y_k\}_1^N$ is a $\delta_{\varepsilon/2}$-net in K (K is compact)
Let $\{x_j\}_{j=1}^{N(\delta_\varepsilon)}$ be the part of $\{x_j\}$ such that

$$\operatorname{dist}\left(\{x_j\}_{j=1}^{N(\delta_\varepsilon)}, y_k\right) \le \delta_{\varepsilon/2} \quad \forall 1\le k\le N.$$

$\Rightarrow$ then $\{x_j\}_{j=1}^{N(\delta_\varepsilon)}$ is a δ_ε-net by triangle inequality. ▨
($\|g_s-g_m\|_{C(K)} \le \varepsilon/3+\varepsilon/3+\varepsilon/3$ for s,m large enough)

Example: $K=[0,1]$, $E_A = \left\{ f \in C[0,1],\ f(0)=0,\ f \text{ is Lipshitz with constant at most } A \right\}$, $A>0$

E_A is compact

i) E_A is bounded in $C[0,1]$:

$$|f(x)| \leq |f(x)-f(0)| \leq A|x| \leq A$$

$$E_A \subset B(0,A)$$

ii) $|f(x)-f(y)| \leq A|x-y| \leq A\delta = \varepsilon$ if $\varepsilon>0$, $\delta := \frac{\varepsilon}{A}$, $x,y \in [0,1]: |x-y| \leq \delta$

$i + ii$ + AA theorem $\Rightarrow E_A$ is precompact

iii) E_A is closed

if $f_n \rightrightarrows f$ in $C(K)$ then $f_n(0) \to f(0) \Rightarrow f(0)=0$

$$|f_n(x)-f_n(y)| \leq A|x-y|$$

$$\downarrow$$

$$|f(x)-f(y)| \Rightarrow f \text{ is } \mathrm{Lip}(A)$$

Example: Let $E = \left\{ \sum_{k\in\mathbb{Z}} c_k e^{2\pi i k x},\ \text{where } c_k \in \mathbb{C} : |c_k| \leq \frac{1}{k^2+1} \right\}$

Then E is compact as well in $C[0,1]$.

i) bdd: $f \in E$. $\|f\| \leq \sum_{k\in\mathbb{Z}} \frac{1}{k^2+1}$

Details: exercise

ii) equicontinuity $F = \underbrace{\sum_{|k|\leq N}}_{\text{Lipshitz with some constant } A_N \text{ - does not depent on } F} + \sum_{|k|>N}$ small if N large

Compact operators: basic properties

Definition: Let X, Y be Banach spaces, $T: X \to Y$ a linear map. T is called **bounded** if $T(B(0,1))$ is a bounded in Y set in Y. T is called **compact** if $T(B(0,1))$ is a precompact set in Y. ($B(0,1) = \{\|x\|_X < 1\}$) bounded linear operator

Some observations:

1) If $S \subset X$ is bdd then $T(S)$ is bdd for any bounded operator
precompact compact

2) T is compact $\Rightarrow$ T is bounded
(precompact sets are bounded)

3) with the norm $\|T\| = \sup\limits_{x \in B(0,1)} \|Tx\|_Y$, the set of bdd linear operators becomes a linear normed space, to be denoted by $B(X,Y)$ or $B(X)$ if $X=Y$.

4) A linear map between Banach spaces X, Y is continuous if and only if it is bounded.
Hint: $\|Tx - Ty\| \leq \|T\| \cdot \|x-y\|$, so bounded operators are Lipshitz.

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Definition: **Banach algebra** is an associative algebra which is a linear space with a norm $\|\cdot\|$ such that it's a Banach space with respect to this norm (it is complete with respect to this norm) and $\|T_1 T_2\| \leq \|T_1\|\|t_2|$ for any elements in this algebra.

Proposition: Let X be a Banach space. Then $B(X,X)$ is a Banach algebra.

Proof: $\alpha_1 T_1 + \alpha_2 T_2 \in \mathcal{B}(X, X) \quad \forall T_1 T_2 \in \mathcal{B}(X, X)$ (proved)

$T_1 \cdot T_2 \in \mathcal{B}(X, X)$, since $\forall x \in X$. $\|T_1 T_2 x\| \overset{}{\leq} \|T_1\| \cdot \|T_2 x\|$

$\Rightarrow \underset{\substack{\sup\limits_{\substack{y \in X \\ \|y\| \leq 1}} \|T_1 T_2 x\|}}{\underset{\shortparallel}{\|T_1\| \cdot \|T_2\|}} \leq \|T_1\| \cdot \|T_2\|$

since $T_1 \in \mathcal{B}(X, X)$ we have $\|T_1 y\| \leq \|T_1\| \|y\| \; \forall y \in X$

We see that $T_1 T_2 \in \mathcal{B}(X, X)$ and $\|T_1 T_2\| \leq \|T_1\| \|T_2\|$.

Now let us prove that $\mathcal{B}(X, X)$ is Banach.

Let us show that $\sum\limits_{n=1}^{\infty} B_k$ converges in $B(X, X)$ if $\sum \|B_k\| \overset{(*)}{<} \infty$.

$T_n = \sum\limits_{k=1}^{N} B_k$, $x \in X$, $\underset{N > M}{\|T_N x - T_M x\|} = \|\sum\limits_{M+1}^{N} B_k x\| \leq \sum\limits_{M+1}^{N} \|B_k\| \cdot \|x\| \xrightarrow[M, N \to \infty]{} 0$,

because of $(*)$.

$\Rightarrow \{T_N x\}_N$ Cauchy in X, but X-banach $\Rightarrow \exists Tx = \lim\limits_{X \to \infty} T_N x$

Moreover, $\|Tx\| = \lim \|T_N x\| \leq \lim\limits_{N \to \infty} \sum\limits_{1}^{N} \|B_k\| \|x\| \leq \left(\sum\limits_{1}^{\infty} \|B_k\|\right) \cdot \|x\|$.

$\Rightarrow T \in \mathcal{B}(X, X)$, $\|T\| \leq \sum\limits_{1}^{\infty} \|B_k\|$

$$\sup_{\substack{x \in X \\ \|x\| < 1}} \|Tx - T_N x\| = \lim_{M \to \infty} \|T_N x - T_M x\| \leq \sup_{\|x\| < 1} \lim_{N \to \infty} \sum_{M+1}^{N} \|B_k\| \cdot \|x\| = \sum_{M+1}^{\infty} \|B_k\| \xrightarrow[(*)]{M \to \infty} 0$$

▨

Proposition: The set $\underset{}{S_\infty(X)} \overset{\shortparallel}{} S_\infty(X, X)$ (index ∞ will be explained later) of all compact operators on X is a two-sided ideal in $\mathcal{B}(X) = \mathcal{B}(X, X)$: $\forall T_1 \in S_\infty(X). \forall T_2 \in \mathcal{B}(X)$. $T_1 T_2 \in S_\infty(X)$ and $T_2 T_1 \in S_\infty(X)$.

Proof: Take $\{x_n\}_1^{\infty}$ s.t. $\|x_n\| \leq 1$, and let us check that there is a subsequence $\{x_{n_k}\}$: $T_1 T_2 x_{n_k}$ converges.

Note that $\{T_2 x_{n_k}\} \subset B_X(0, \|T_2\|)$. T_1 takes $\mathcal{B}_X(0, \|T_2\|)$ into a precompact subset of $X \Rightarrow \exists \{T_1 T_2 x_{n_k}\}$ - convergent subsequence

Now let's consider $\{T_2 T_1 x_n\}$. Note that $\{T_1 x_{n_k}\}$ - convergent subsequence ($T_1 \in S_\infty(X)$). Then $\{T_2 T_1 x_{n_k}\}$ converges, since T_2 is continuous.

▨

Proposition: $S_\infty(X,Y)$ is a closed subset in $\mathcal{B}(X,Y)$, i.e. $T_n(X,Y)$, $T_n \longrightarrow T$ in $\mathcal{B}(X,Y) \Rightarrow T \in S_\infty(X,Y)$.

Proof: Let's find a finite ε-net in $T(B_X[0,1])$. Take finite $\varepsilon/3$-net for $T_n B_X(0,1)$ for $n : \|T-T_n\| \le \varepsilon/2$; denote it by $\{y_k\}_{k=1}^N$, then

$$\|Tx - Tx_k\| \le \underbrace{\|Tx - T_nx\|}_{A} + \underbrace{\|T_nx - T_nx_k\|}_{B} + \underbrace{\|T_nx_k - Tx_k\|}_{C} \le \varepsilon/3 + \varepsilon/3 + \varepsilon/3 \le \varepsilon.$$

for any $1 \le k \le N$
so choose k : $B \le \varepsilon/2$
and note $A \le \|T-T_n\| \le \varepsilon/3$
for every $x \in B_X(0,1)$
$C \le \varepsilon/3$

Corollary: If T is a limit of finite-rank operators in $\mathcal{B}(X,Y)$, then $T \in S_\infty(X,Y)$.

Proof: Since finite-rank operators are in $S_\infty(X,Y)$, we have $T \in S_\infty(X,Y)$ by the previous proposition. ☒

Remark: At a general Banach space $\exists T \in S_\infty(X,Y)$ such that $\nexists (T_n)_n : \operatorname{rank} T_n < \infty$ and $\|T - T_n\| \longrightarrow 0$.

Definition: Let X be a Banach space. $\{e_k\}_{k=1}^\infty$ is a Schauder basis if $\forall x \in X.\ \exists! \{c_k(x)\}_{k=1}^\infty$ such that $x = \sum_{k=1}^\infty c_k(x) e_k$, where the series converges in X.

Theorem: Let X be a Banach space with Schauder basis. Then $T \in S_\infty(X) \Leftrightarrow \exists T_n .\ \operatorname{rank} T_n \le n$ and $\|T - T_n\| \to 0$.
(here $\operatorname{rank} S = \dim S(X)$ $\forall S \in \mathcal{B}(X)$)

Proof: $(\Leftarrow)$: we already know

(⇐): Let $T \in S_\infty(X)$, and let $P_n : x \longmapsto \sum_{k=1}^{\infty} c_k(x) e_k$.

<u>P is linear</u>: $\forall \alpha, \beta \in \mathbb{C}, \forall x, y \in X.\ P_n(\alpha x + \beta y) = \alpha P_n(x) + \beta P_n(y)$?

$$\left.\begin{array}{l} \text{If } x = \sum c_k(x) e_k \\ \quad y = \sum c_k(y) e_k \end{array}\right\} \Rightarrow \alpha x + \beta y = \sum_{k=1}^{\infty} (\alpha c_k(x) + \beta c_k(y)) e_k$$

$$\alpha x + \beta y \overset{\nwarrow}{=} \sum_{k=1}^{\infty} c_k(\alpha x + \beta y) e_k$$

by uniqueness ↘ by def. of Schauder basis

$$c_k(\alpha x + \beta y) = \alpha c_k(x) + \beta c_k(y) \quad \forall k$$

Then $P_n(\alpha x + \beta y) = \sum_{n=1}^{n} c_k(\alpha x + \beta y) e_k = \sum_{k=1}^{k} \alpha c_k(x) e_k + \sum_{n=1}^{k} \beta c_k(y) e_k$

$$= \alpha P_n(x) + \beta P_n(y) \quad \Rightarrow P_n \text{ linear}$$

Note that $T_n := P_n T$ are such that $\operatorname{rank}(T_n) \leq n$ because $\dim P_n T(X) \leq \dim P_n(X) \leq n$.

It remains to show that $T_n \longrightarrow T$ in $\mathcal{B}(X)$. Since T is compact, $\forall \varepsilon > 0.\ \exists \{x_k\}_{k=1}^{N}$ such that $\|x_k\| \leq 1\ \forall k$ and $\{T x_k\}_{k=1}^{N}$ is a ε-net in $T(B_X(0,1))$. Now take $x \in B_X(0,1)$ and write $\|Tx - T_n x\| \leq \|Tx - Tx_k\| + \|Tx_k - T_n x\| + \|T_n x_k - T_n x\|$

$$\leq \underbrace{\|Tx - Tx_k\|}_{\leq \varepsilon \text{ for some } k} + \underbrace{\|Tx_k - P_n T x_k\|}_{\substack{\leq \varepsilon \text{ if } n \\ \text{large enough} \\ \text{for any fixed } k}} + \underbrace{\|P_n T x_k - P_n T x\|}_{\leq \|P_n\| \cdot \|Tx_k - Tx\|}$$

$$\leq \varepsilon + \varepsilon + \underbrace{\sup \|P_n\|}_{< \infty} \varepsilon$$

$\sup \|P_n\| < \infty$ by Banach-Sceinhaus theorem on uniform point-wise convergence.

$\|T - T_n\| \leq \varepsilon(2 + \sup \|P_n\|)$ for n large enough ▨

Theorem [Banach-Steinhaus]: Assume that $\{T_n\}_{n=1}^{\infty} \subset \mathcal{B}(X)$ where X is a Banach space such that

$$\sup_n \|T_n x\| \leq C(x) < \infty$$

local information
~ uniform estimate

for every $x \in X$. Then $\sup_n \|T_n\| < \infty$. In particular, one can take C in place of $C(x)$.

Remark: In our situation $\sup_{1 \leq n < \infty} \|P_n\| < C(x) < \infty$ because $P_n x \longrightarrow x$ in X.

Banach adjoint operators

$\boxed{\text{October 15, 2025}}$

Definition: Let X be a Banach space. Then $X^* = \mathcal{B}(X, \mathbb{C})$ is called the dual space to X. The elements of X^* are called functionals.

$$= \left\{ \phi : X \to \mathbb{C} \;\middle|\; \begin{matrix} \phi \text{ linear} \\ \|\phi\| < \infty \end{matrix} \right\}$$

Examples: (can ignore, if one does not know measure theory)

i) $\underset{1 \leq p < \infty}{L^p(\mu)} = \left\{ f : S \longrightarrow \mathbb{C} \;\middle|\; \begin{matrix} f \text{ is measurable with} \\ \text{respect to } \sigma\text{-algebra} \\ \text{of } \mu \end{matrix}, \int_S |f|^p \, d\mu < \infty \right\} \Big/ \begin{matrix} f = g \\ \text{if } f(x) = g(x) \\ \text{for } \mu\text{-a.e. } x \in S \end{matrix}$

$$\|f\|_{L^p(\mu)} = \left(\int_X |f|^p \, d\mu \right)^{1/p}$$

$$(L^p(\mu))^* = L^q(\mu) \quad \text{where} \quad \frac{1}{p} + \frac{1}{q} = 1$$

ii) $\underset{1 \leq p < \infty}{\ell^p(\mathbb{Z})} = \left(\left\{ \{x_k\}_{k \in \mathbb{Z}} \;\middle|\; \sum_{k \in \mathbb{Z}} |x_k|^p < \infty \right\}, \|x\|_{\ell^p(\mathbb{Z})} = \left(\sum |x|^p \right)^{1/p} \right)$

$$\ell^p(\mathbb{Z})^* = \ell^q(\mathbb{Z}), \quad \text{where} \quad \frac{1}{p} + \frac{1}{q} = 1$$

In these examples, the following identification is assumed:

i) $g \in L^q(\mu) \leftrightsquigarrow \Phi_g : f \longmapsto \int_S f g \, d\mu$, $\Phi_g : L^p(\mu) \longrightarrow \mathbb{C}$

ii) $\{y_k\}_{k=1}^{\infty}$ in $\ell^q(\mathbb{Z}) \leftrightsquigarrow \Phi_{\{y_k\}} : \{x_k\} \longmapsto \sum_{k\in\mathbb{Z}} x_k y_k$

$\Phi_{\{y_k\}} : \ell^p(\mathbb{Z}) \to \mathbb{C}$

Remark: i) is non-trivial measure theory

More examples:

iii) $c_0(\mathbb{Z}) = \{ \{x_k\}_{k\in\mathbb{Z}} \mid x_k \to 0 \text{ as } |k| \longrightarrow \infty \}$

$c_0^*(\mathbb{Z}) = \ell^1(\mathbb{Z})$ same identification

(Hausdorff is actually sufficient ⤳ hard)

iv) Let K be a compact metric space, and $X = C(K)$. Then $X^* = \mathcal{M}(K)$.

set of cont. maps $(K, \delta) \to \mathbb{C}$

$$\mathcal{M}(K) = \left\{ \begin{array}{l} \text{the set of Borel} \\ \text{measures on } K \\ \text{(complex valued)} \end{array} \;\middle|\; |\mu|(K) = \sup_{\substack{K = \cup E_n \\ E_n \cap E_j = \emptyset \\ n \neq j}} \sum_{n\in\mathbb{Z}} |\mu(E_n)| < \infty \right\}$$

$\|\mu\| = |\mu|(K)$

Riezs-Markov representation theorem

Here $\mu \in \mathcal{M} \overset{(*)}{\leftrightsquigarrow} \Phi_\mu : f \longrightarrow \int_K f \, d\mu$

We can also define ℓ^p, L^p for $p = \infty$:

- $\ell^\infty(\mathbb{Z}) := \{ \{x_k\} \subset \mathbb{C} : \sup_{k\in\mathbb{Z}} |x_k| < \infty$
- $L^\infty(\mu) := \{ f : \_\_\_ , \_\_\_ : \operatorname{ess\,sup} |f| < \infty \}$

Remark: If $1 < p < \infty$ then $(L^p)^* = L^q$, $(L^q)^* = L^p$

But for $p=1$ $\quad (L^1)^* = L^\infty, \quad (L^\infty)^* \neq L^1$

$\ell^1(\mathbb{Z})^* = \ell^\infty(\mathbb{Z})$, but $(\ell^\infty(\mathbb{Z}))^* \neq \ell^1(\mathbb{Z})$

Definition: Let X, Y be Banach spaces, $T \in \mathcal{B}(X,Y)$. Then $T^* \in \mathcal{B}(Y^*, X^*)$ is defined by

$$T^* : \underset{\in Y^*}{\Psi} \longmapsto \Big(\underset{\in X^*}{(T^*\Psi)} : x \mapsto \underset{\psi(Tx)}{\underset{\shortparallel}{\langle Tx, \psi \rangle}} \Big),$$

where $\langle x, \phi \rangle = \phi(x)$ for $x \in X$, $\phi \in X^*$.

Remark: $\langle Tx, \psi \rangle = \langle x, T^*\psi \rangle$ $\rightarrow$ this formula is equivalent to the definition of T^*

Remark: Operation that sends x, ϕ into $\phi(x) = \langle x, \phi \rangle$ for $x \in X$, $\phi \in X^*$ is called a pairing of Banach spaces X, X^*.

Example: For $\varphi \in C[0,1]$, μ on $[0,1]$ then the pairing is $\langle \varphi, \mu \rangle = \int_0^1 \varphi \, d\mu$, see $(\star)$.

Theorem: Let X, Y be Banach spaces, $T \in \mathcal{B}(X,Y)$. Then the map $T^* : Y^* \longrightarrow X^*$ defined by $\langle x, T^*\psi \rangle := \langle Tx, \psi \rangle$, $x \in X$, is an element of $\mathcal{B}(Y^*, X^*)$.

$(\Leftrightarrow \ (T^*\psi)(x) = \psi(Tx))$

Lemma 1 [Hahn-Banach theorem]: Let X be a Banach space, $E \subset X$ - subspace in X, $\phi_0 : E \to \mathbb{C}$ is linear and bdd ($\Leftrightarrow \phi_0 \in E^*$). Then $\exists \, \phi \in X^*$ such that $\phi|_E = \phi_0$ and $\|\phi\| = \|\phi_0\|$.

Lemma 2 ["Sufficient amount of functionals"]:

Let $x \in X$, then $\|x\| = \sup\limits_{\|\Phi\|\le 1} |\Phi(x)|$.

Proof: $|\Phi(x)| \le \|\Phi\|\cdot\|x\| \le \|x\|$, so $\|x\| \ge \sup\limits_{\|\Phi\|\le 1} |\Phi(x)|$

To prove "$\le$", define $E = \operatorname{span}\{x\} = \{\lambda x, \lambda \in \mathbb{C}\}$,
$\Phi_0 : y \longrightarrow c_y$ if $y = c_y x \in E$.
Assume that $\|x\| = 1$, then $\|\Phi_0\|_{E^*} = \sup\limits_{\|y\|\le 1} |c_y| =$ (from (**), $|c_y| = \|y\|$ if $\|x\| = 1$) $= \sup\limits_{\|y\|\le 1} \|y\| = 1$.

Hahn-Banach theorem $\Rightarrow \exists\, \tilde{\Phi}_0 \in X^* : \|\tilde{\Phi}_0\| = 1$, $\Phi|_E = \Phi_0$.
In particular $\sup\limits_{\|\Phi\|\le 1} |\Phi(x)| \ge |\tilde{\Phi}| = |\Phi_0(x)| = 1 = \|x\|$.

We have proved "$\le$" in the case where $\|x\| = 1$.
The general case follows from consideration of $\frac{x}{\|x\|}$ in place of X.

October 21, 2025

We are proving that $T \in \mathcal{B}(X,Y) \Rightarrow T^* \in \mathcal{B}(Y^*, X^*)$ and $\|T\| = \|T^*\|$.

Let $T \in \mathcal{B}(X,Y)$, consider

$$
\begin{aligned}
\|T^*\| &= \sup_{\substack{\psi \in Y^* \\ \|\psi\|\le 1}} \|T^*\psi\|_{X^*} = \sup_{\substack{\psi \in Y^* \\ \|\psi\|\le 1}} \sup_{\substack{x \in X \\ \|x\|\le 1}} |(T^*\psi)(x)| \\
&= \sup_{-''-}\sup |\langle x, T^*\psi\rangle| \\
&= \sup\sup |\langle Tx, \psi\rangle| \\
&= \sup_{\substack{\psi \in Y^* \\ \|\psi\|\le 1}} \sup_{\substack{x \in X \\ \|x\|\le 1}} |\psi(Tx)| \\
&= \sup_{\substack{x \in X \\ \|x\|\le 1}} \boxed{\sup_{\substack{\psi \in Y^* \\ \|\psi\|\le 1}} |\psi(Tx)|} \overset{\text{Lemma "sufficient amount of functionals"}}{=} \|Tx\|
\end{aligned}
$$

$$= \sup_{\|x\| \le 1} \|Tx\| = \|T\| < \infty$$

The claim follows. ▨

Corollary: $T \in \mathcal{B}(X,Y)$ is invertible $(\exists T^{-1} \in \mathcal{B}(Y,X))$ iff $T^* \in \mathcal{B}(Y^*, X^*)$ is invertible $(\exists\, (T^*)^{-1} \in \mathcal{B}(X^*, Y^*))$.

We prove just $\Rightarrow$.

Proof: Assume that T is invertible $\Leftrightarrow$ $T^{-1}T = I_X$

$$TT^{-1} = I_Y$$

Let's take adjoint operators and see:

$$\left.\begin{array}{l} (T^{-1}T)^* = (I_X)^* \\ (TT^{-1})^* = (I_Y)^* \end{array}\right\} \Leftrightarrow \left\{\begin{array}{l} T^*(T^{-1})^* = I_X^* \\ (T^{-1})^* T^* = I_Y^* \end{array}\right.$$

Exercise: $(AB)^* = B^*A^*$

It remains to check that $I_X^* = I_{X^*}$, $I_Y^* = I_{Y^*}$. Then, by the previous theorem, $(T^{-1})^* \in \mathcal{B}(X^*, Y^*)$, hence T^* is invertible and its bounded inverse is $(T^*)^{-1} = (T^{-1})^*$.

Let's check that $I_X^* = I_{X^*}$. Take $\Phi \in X^*$, $x \in X$.

$$(I_X^* \Phi)(x) = \langle x, I_X^* \Phi \rangle = \langle I_X x, \Phi \rangle = \langle x, \Phi \rangle = \Phi x$$

$$(I_{X^*} \Phi(x)) \underset{\text{def}}{=} (\Phi)(x) = \Phi(x).$$

Similarly, $I_Y^* = I_{Y^*}$. ▨

The „pairing notation" is often not used in literature, but it is very usefull to not make mistakes.

Theorem [Schauder]: We have $T \in S_\infty(X,Y) \Leftrightarrow T^{*}_{\infty}(Y^*, X^*)$.

Proof: We will prove just "$\Rightarrow$".

Consider $K = \overline{TB_X(0,1)}$ – a compact set. Let $C(K)$ be the Banach space of continuous functions on K with

$$\|f\|_{C(K)} = \max_{s\in K}|f(s)|, \quad f: K \to \mathbb{C}$$

Let $E := \{\psi \in Y^* \mid \|\psi\|_{Y^*} \le 1, \ \psi \text{ is considered as a function on } K\}$

$K \subset Y$, K metric space with respect to the metric $\rho(y_1,y_2) = \|y_2 - y_1\|_Y$

So, $E \subset C(K)$ and we claim that E is precompact.

1) uniform boundedness:

$$\psi \in E \Rightarrow \|\psi\|_{C(K)} = \max_{s\in K}|\psi(s)| = \max_{s \in \overline{TB_X(0,1)}} |\psi(s)| \overset{\psi \text{ cont.}}{=} \sup_{x \in B_X(0,1)} |\psi(Ts)|$$

$$\le \|\psi\| \sup_{\|x\|\le 1} \|Tx\| \le \|\psi\| \cdot \|T\| \le \underbrace{\|T\| < \infty}_{\text{does not depent on } \psi}$$

2) Equicontinuity: take $s_1, s_2 \in K$, let's estimate

$$|\psi(s_1) - \psi(s_2)| = |\psi(s_1 - s_2)| \le \|\psi\| \cdot \|s_1 - s_2\| \le \|s_1 - s_2\|,$$

so maps from E are Lipschitz with constant 1, hence equicontinuous.

$\Rightarrow$ By Arzela-Askoli theorem, E is precompact.

We are now ready to prove $T^* \in S_\infty(Y^*, X^*)$. For this, we need to check that if $\{\psi_n\}$ is a sequence in $B_{Y^*}(0,1)$, then $\exists \{\psi_{n_k}\}$ such that $T^*\psi_{n_k}$ converges in X^*. So, take $\{\psi_n\} \subset B_{Y^*}(0,1)$ and consider it as elements $E \subset C(K)$.

Let $\{\psi_{n_k}\}$ be such that $\psi_{n_k} \xrightarrow{(*)} \psi$ in $C(K)$. $\left(\begin{matrix}E \text{ is}\\ \text{precompact!}\end{matrix}\right)$

Let's prove that $\{T^*\psi_{n_k}\}$ is Cauchy in X^*, then the theorem will follow.

Take $x \in X$, $\|x\| \leq 1$, and consider

$$\begin{aligned} \|(T^*\psi_{n_k})(x) - (T^*\psi_{n_j})(x)\| &= \|\langle x, T^*\psi_{n_k}\rangle - \langle x, T^*\psi_{n_j}\rangle\| \\ &= \|\langle Tx, \psi_{n_k}\rangle - \langle Tx, \psi_{n_j}\rangle\| \\ &= \|\psi_{n_k}(Tx) - \psi_{n_j}(Tx)\| \\ &\leq \sup_{s \in K} |\psi_{n_k}(s) - \psi_{n_j}(s)| \\ &= \underbrace{\|\psi_{n_k} - \psi_{n_j}\|_{C(K)}}_{\overset{''}{\varepsilon_{kj}} \text{ - does not depend on } x} \longrightarrow 0 \text{ by } (*). \end{aligned}$$

$$\Rightarrow \|T^*\psi_{n_k} - T^*\psi_{n_j}\| \leq \varepsilon_{k,j} \longrightarrow 0.$$

▨

Fredholm alternative

Example: Consider the equation $f(t) - \int_0^1 e^{t-s} f(s)\,ds \overset{(**)}{=} g(t)$ in $L^2[0,1]$.

Question: For which $g \in L^2[0,1]$ do we have a solution $f \in L^2[0,1]$?

Observation: g has to satisfy $\int_0^1 e^{-t} g(t)\,dt \overset{(***)}{=} 0$

$\int_0^1 c\,dt = c$

Indeed, $\int_0^1 e^{-t} g(t)\,dt = \int_0^1 e^{-t} f(t)\,dt - \int_0^1 \cancel{e^{-t}} \left(\int_0^1 \cancel{e^{t}}{}^{-s} f(s)\,ds \right) dt = 0$

It is not clear so far if there are other restrictions.

Theorem [Fredholm alternative]: Let X be a Banach space, $T = I - K$ for $K \in S_\infty(X,X)$. Then

$$\operatorname{Ran} T = \{x \in X \mid \langle x, \Phi\rangle = 0 \ \forall \Phi \in \ker T^*\}.$$

In other words, either:

(1) $\ker T^* = \{0\}$ and the equation $Tf = g$ has solution $\forall g \in X$.

or

(2) $\operatorname{Ker} T^* \neq \{0\}$ and the equation $Tf = g$ has solutions only for g s.t. $\langle g, \Phi\rangle = 0 \ \forall \phi \in \ker T^*$.

Let's complete the consideration of the example: we need to check that $K: f \longrightarrow \int_0^1 e^{t-s} f(s)ds$ is compact (exercise) and find $\operatorname{Ker}T^*$.

$$\phi \in \operatorname{Ker}T^* \iff T^*\varphi = 0$$

Adjoint operator T^* is defined by

$$\langle Tf, g\rangle = \langle f, T^*g\rangle \qquad f,g \in L^2[0,1]$$

$(L^2[0,1]^* = L^2[0,1])$

$$\iff \langle f - \int_0^1 e^{t-s} f, g\rangle = \int_0^1 f(t)g(t)dt - \int_0^1\int_0^1 e^{t-s} f(s)ds\, g(t)dt$$

$$= \int_0^1 f(t)g(t)dt - \int_0^1 f(s)\left(\int_0^1 e^{t-s} g(t)dt\right)ds$$

$$= \langle f, g - \int_0^1 e^{t-s} g(t)dt\rangle_{L^2[0,1]}$$

$(T^*g): s \longmapsto g(s) - \int_0^1 e^{t-s} g(t)dt, \quad s \in [0,1].$

$T^*g = 0 \iff g(s) = \int_0^1 e^{t-s} g(t)dt$ a.e. on $[0,1]$

$\iff e^s g(s) = \underbrace{\int_0^1 e^t g(t)dt}_{\text{constant}}$ for almost every $s \in [0,1]$

$\Rightarrow$ So, $\ker T^* = \{c \cdot e^{-s}, c \in \mathbb{C}\}$, $\dim(\operatorname{Ker}T^*) = 1$.
By Fredholm theorem, equation $(**)$ is solvable $\iff$ $\forall c \in \mathbb{C}.\ \langle g, c\cdot e^{-s}\rangle = 0 \iff \int_0^1 g(s)e^{-s}ds = 0$, which is $(***)$.

Preliminaries

Lemma [almost orthogonality in Banach spaces]: Let X be a Banach space, $E \subsetneq X$ - a linear closed subspace, $\varepsilon > 0$. Then $\exists x_0 \in X$ such that $\|x_0\| = 1$, $\operatorname{dist}(x_0, E) \geq 1 - \varepsilon$.

Proof: Since $E \neq X$, then $\exists\, \tilde{x}_0 \in X \setminus E$. Since E is closed, we have $\operatorname{dist}(\tilde{x}_0, E) = \delta > 0$ for some $\delta > 0$. Now consider $\tilde{y}_0 \in E$ such that $\delta \leq \|\tilde{x}_0 - \tilde{y}_0\| \leq (1+\eta)\delta$ for some $\eta \in (0,1)$.

Now let $x_\eta := \dfrac{\tilde{x}_0 - \tilde{y}_0}{\|\tilde{x}_0 - \tilde{y}_0\|}$, $\|x_\eta\| = 1$.

$$\operatorname{dist}(x_\eta, E) = \frac{1}{\|\tilde{x}_0 - \tilde{y}_0\|} \operatorname{dist}(\tilde{x}_0 - \tilde{y}_0, E)$$

$$= \frac{1}{\|\tilde{x}_0 - \tilde{y}_0\|} \operatorname{dist}(\tilde{x}_0, E)$$

$$= \frac{\delta}{\|\tilde{x}_0 - \tilde{y}_0\|} \geq \frac{1}{1+\eta}$$

Choosing η so that $\frac{1}{1+\eta} = 1 - \varepsilon$, we are done.

October 22, 2025

Lemma: Let X be a Banach space. Then $I : x \mapsto x$ is compact on $X \iff \dim X < \infty$.

Proof: $\dim X < \infty \Rightarrow I \in S_\infty(X)$ – we already know

$I \in S_\infty(X) \Rightarrow \dim X < \infty$:

Suppose $\dim X = +\infty$, find a sequence $\{e_n\}$: $\|e_n\| = 1$ $\forall n \in \mathbb{N}$

$e_1 \in X$ – arbitrary

e_2 : $\operatorname{dist}(e_2, \operatorname{span}\{e_1\}) \geq 1/2$

e_3 : $\operatorname{dist}(e_3, \operatorname{span}\{e_1, e_2\}) \geq 1/2$

e_4 : etc

existence of $\{e_n\}$ follows from previous lemma, because $\operatorname{span}\{e_1, \dots, e_k\} \neq X$ $\forall k \in \mathbb{N}$

Then $\{e_n\} \subset B_X[0,1] = I(B_X[0,1])$ but there is no convergent subsequence, because $\|e_k - e_j\| \geq 1/2$ $\forall k, j$. ▨

Lemma: Let X be a Banach space, $K \in S_\infty(X)$, $T = I - K$.
Then: 1) $\dim(\operatorname{Ker} T) < \infty$.
2) $\operatorname{Ran} T$ is closed in X. [closed range Lemma]

Proof: 1): We have $I|_{\operatorname{Ker}T} = \underbrace{(I-K)|_{\operatorname{Ker}T}}_{0} + \underbrace{K|_{\operatorname{Ker}T}}_{\in S_\infty(\operatorname{Ker}T,\, S)}$

$$I|_{\operatorname{Ker}T} \in S_\infty(\operatorname{Ker}T, X) \Rightarrow I \in S^\infty(\operatorname{Ker}T) \Rightarrow \dim(\operatorname{Ker}T) < \infty$$

2): The statement is equivalent to the fact that if $\{x_n\} \subset X$ s.t. $Tx_n \longrightarrow y$ in X then $\exists x \in X.\ Tx = y$.

2.a) Let $\{x_n\}: \|x_n\| \le C \ \forall n$.
Then $(I-K)(x_n) \longrightarrow y$, $(I-K)(x_{n_k}) \xrightarrow{(*)} y$
For every subsequence x_{n_k}

Let's choose x_{n_k}: Kx_{n_k} converges to $z \in X$
(use $K \in S_\infty(X)$)

Then $x_{n_k} \longrightarrow y + z$ by $(*)$, take $x = y + z$:

$T(y+z) = \lim\limits_{k\to\infty} Tx_{n_k} = y$, so $Tx = y$. ✓

2.b) $\operatorname{dist}(x_n, \operatorname{Ker}T) \le C \quad \forall n \in \mathbb{Z}$
Take $\tilde{x}_n := x_n - w_n$, where $w_n \in \operatorname{Ker}T : \|\tilde{x}_n) \le 2C$.
We have $\lim\limits_{n\to\infty} T\tilde{x}_n = \lim\limits_{n\to\infty} Tx_n$ by step 2a) $\exists \tilde{x}: T\tilde{x} = y$ ✓

2.c) $\operatorname{dist}(x_n, \operatorname{Ker}T) \longrightarrow +\infty$. Let us show that this situation does not occur. Suppose the converse:
Consider $\tilde{x}_n = x_n - \underset{\operatorname{Ker}T}{w_n}$: $\underbrace{\operatorname{dist}(x_n, \operatorname{Ker}T) \le \|\tilde{x}_n\| \le 2\operatorname{dist}(x_n, \operatorname{ker}T)}_{(**)}$

For $z_n = \frac{\tilde{x}_n}{\|\tilde{x}_n\|}$ we have $Tz \longrightarrow 0$.

$$Tz_n = T\frac{\tilde{x}_n}{\|\tilde{x}_n\|} = T\frac{x_n}{\|\tilde{x}_n\|} = \frac{\boxed{Tx_n} \to y}{\boxed{\|\tilde{x}_n\|} \to +\infty}$$

$\Rightarrow Tx_n$ is bdd in X

$$\Rightarrow \|Tz_n\| \le \frac{2\|y\|}{\|\tilde{x}_n\|} \longrightarrow 0 \quad \text{For } n \text{ large enough}$$

At the same time, $Tz_n = z_n - Kz_n$

$\Rightarrow \exists \{z_{n_k}\}$ s.t. $\{Kz_{n_k}\}$ converges to some $z \in X$

$$\Rightarrow z_{n_k} = \underbrace{Tz_{n_k}}_{\to 0} + \underbrace{Kz_{n_k}}_{\to z} \longrightarrow z$$

We have $Tz = 0$ $\left(= \lim Tz_{n_k} = \lim Tz_n = 0\right)$

$$\begin{aligned}\Leftrightarrow z \in \ker T,\ 0 = \operatorname{dist}(z, \ker T) &= \\ &= \lim_{k\to\infty} \operatorname{dist}(z_{n_k}, \ker T) \\ &= \lim_{k\to\infty} \operatorname{dist}\left(\frac{\tilde{x}_{n_k}}{\|\tilde{x}_{n_k}\|}, \ker T\right) \\ &= \lim \frac{\operatorname{dist}(\tilde{x}_{n_k}, \ker T)}{\|\tilde{x}_{n_k}\|} \\ &= \lim \frac{\operatorname{dist}(x_{n_k}, \ker T)}{\|\tilde{x}_{n_k}\|} \\ &\overset{(**)}{\ge} \frac{1}{2} \rightsquigarrow \text{contradiction}\end{aligned}$$

▨

Lemma: Let X be a Banach space, $T \in \mathcal{B}(X)$: $\ker(T) = \{0\}$ and $T^{k+1}X = T^k X$ for some $k \ge 0$. Then $\operatorname{Ran} T = X$.

Proof: We need to prove that $\forall a \in X. \exists \tilde{a} \in X. T\tilde{a} = a$.
We know that: $T^{k+1}a = T^k\tilde{a}$ for every a and some $\tilde{a}$ depending on a.

$\ker T^k \ne \{0\} \Rightarrow \ker T \ne \{0\}$

$$\Rightarrow T^k(a - T\tilde{a}) = 0 \Rightarrow a - T\tilde{a} = 0 \Rightarrow a = T\tilde{a}.$$

▨

October 28, 2025

Theorem [Fredholm]: Let X be a Banach space. $K \in S_\infty(X)$, $T = 1 - K$. Then TFAE:

1) T is invertible in $\mathcal{B}(X)$ 2) $\operatorname{Ker} T = \{0\}$ 3) $\operatorname{Ran} T = X$

1') T^* is invertible in $\mathcal{B}(X)$ 2') $\operatorname{Ker} T^* = \{0\}$ 3') $\operatorname{Ran} T^* = X^*$

Proof: We will prove $2 \Rightarrow 3 \Rightarrow 2' \Rightarrow 3' \Rightarrow 2$, $1 \Rightarrow 1' \Rightarrow 2'$, $2 \& 3 \Rightarrow 1$

(2) ⇒ (3): If $T^k X = T^{k+1} X$ for some k, we are done by the lemma. Define $X_k := T^k X$, $k \geq 0$, and note that $X_0 \supset X_1 \supset X_2 \supset X_3 \dots$ Assume that all inclusions are strict, i.e. $X_k \supsetneq X_{k+1}$ $\forall k$. The subspaces X_k are closed by the closed range lemma (by induction). By the almost orthogonality lemma: $\exists \{y_k\}_0^\infty$ s.t.

i) $y_k \in X_k$ $\forall k$ ii) $\|y_k\| = 1$ $\forall k$ iii) $\operatorname{dist}(y_k, X_{k+1}) \geq 1/2$

Since K is compact $\{Ky_k\}$ contains a convergent subsequence $\{Ky_{k_j}\}_{j=1}^\infty$. On the other hand, if $j < m$

$$Ky_{k_j} - Ky_{k_m} = \left(Ky_{k_j} - y_{k_j}\right) - \left(Ky_{k_m} - y_{k_m}\right) + y_{k_j} - y_{k_m}$$

$$= y_{k_j} - \underbrace{y_{k_m} - \left(Ty_{k_j} - Ty_{k_m}\right)}_{\in X_{k_j+1}:}$$

Since $y_{k_m} \in X_{k_m}$, $j < m$, $k_j < k_m$, $k_j + 1 \leq k_m$, $X_{k_m} \subset X_{k_j+1}$

$\Rightarrow y_{k_m} \in X_{k_j+1}$; $Ty_{k_m} \in T X_{k_j+1} \subset X_{k_j+1}$; $Ty_{k_j} \in T(X_{k_j}) = X_{k_j+1}$

$\Rightarrow Ky_{k_j} - Ky_{k_m} = y_{k_j} + R$, $R \in X_{k_j+1}$. Since $\|y_{k_j} + R\| \geq 1/2$ by (iii), we get a contradiction ($\{Ky_{k_j}\}$ is not Cauchy).

(3) ⇒ (2'): Take $\phi \in \operatorname{Ker} T^*$. We have

$T^*\phi = 0 \Leftrightarrow \langle x, T^*\phi \rangle = 0$ $\forall x \in X \Leftrightarrow \langle Tx, \phi \rangle = 0$ $\forall x \in X \Leftrightarrow$
$\Leftrightarrow \langle y, \phi \rangle = 0$ $\forall y \in X$ (by 3) $\Leftrightarrow \phi = 0$

(2') ⇒ (3'): (the same as $2 \Rightarrow 3$ using Schauder's theorem)

(3') ⇒ (2): Assume that $\operatorname{Ran} T^* = X^*$ and take $x \in \operatorname{Ker} T$.
We have $Tx = 0 \iff \langle Tx, \phi \rangle = 0 \ \forall \phi \in X^* \iff \langle x, T^*\phi \rangle \ \forall \phi \in X^*$
$\iff \langle x, \psi \rangle = 0 \ \forall \psi \in X^*$ (by 3') $\iff x = 0$ lemma on sufficient amount of functionals

Conclusion: $(2) \iff (3) \iff (2') \iff (3')$

(1) ⇒ (1'): We already know for arbitrary $T \in \mathcal{B}(X)$.

(1') ⇒ (2'): Invertible operators are injective.

$(2') \iff (2), (3)$

(2 & 3) ⇒ (1): This holds for every $T \in \mathcal{B}(X)$ by the following fundamental theorem from general F.A.:

Theorem [linear mapping theorem]: Let X, Y be a Banach space, $T: X \longrightarrow Y$ - linear bijection. Then $T \in \mathcal{B}(X,Y) \iff T^{-1} \in \mathcal{B}(Y,X)$.

Check injectivity: $Tx_1 = Tx_2 \iff T(x_1 - x_2) = 0 \iff x_1 - x_2 = 0$ (by 2) $\iff x_1 = x_2$
surjectivity: $\operatorname{Ran} T = X$ (by 3)

▨

Remark: If $T = 1 - K$, $K \in S_\infty(X)$, and $\operatorname{Ker} T = \{0\}$ then the equation $Tx = y$ has a unique solution for every $y \in X$.

Remark: $\operatorname{Ker} T = \{0\} \iff \operatorname{Ker} T^* = \{0\}$, so we have proved half of Fredholm Alternative.

Theorem: Let X be a Banach space, $T \in \mathcal{B}(X)$. Then
$\overline{TX} = \{ x \in X \mid \langle x, \phi \rangle = 0 \ \forall \phi \in \operatorname{Ker} T \}$.

Remark: This implies the other half of Fredholm alternative since $\overline{TX} = TX$ for $T = 1 - K$, $K \in S_\infty(X)$.

Lemma [seperation lemma]: Let Y be a Banach space, $Y_0 \subsetneq Y$ - a closed subspace, then $\exists \phi \in Y^* : \phi|_{Y_0} = 0$, $\phi(y) \neq 0$ for some $y \in Y \setminus Y_0$.

Proof: Take $y \in Y \setminus Y_0$, define $\phi_0 : \operatorname{span}\{y, Y_0\} \longrightarrow \mathbb{C}$ by $\phi_0(cy + y_0) \longmapsto c$, for $c \in \mathbb{C}$, $y_0 \in Y_0$.

1) $\{cy, y_0 \mid c \in \mathbb{C}, y_0 \in Y_0\} = \operatorname{span}\{y, Y_0\}$ - clear ✓

2) $\underbrace{cy + y_0}_{e} = \underbrace{\tilde{c}y + \tilde{y}_0}_{\tilde{e}} \Leftrightarrow (c - \tilde{c})y = y_0 - \tilde{y}_0 \in Y_0 \Leftrightarrow c = \tilde{c} \Rightarrow \phi(e) = \phi(\tilde{e})$

$\Rightarrow$ correctness OK

3) ϕ_0 is linear - clear

4) $\phi_0|_{Y_0} = 0$ $(c = 0 \text{ on } Y)$

5) $|\phi_0(cy + y_0)| \overset{?}{\leq} A\|cy + y_0\| \quad \forall c, y_0$

$$|\phi_0(cy + Y_0)| = |c| = (\operatorname{dist}(y, \underbrace{Y_0}_{\text{closed}}))^{-1} \cdot |c| \cdot \operatorname{dist}(y, Y_0)$$

$$= \operatorname{dist}(y, Y_0)^{-1} \cdot \operatorname{dist}(|c|y, Y_0) = \operatorname{dist}(y, Y_0)^{-1} \cdot \operatorname{dist}(cy, Y_0)$$

$\uparrow$

$|c|y = \alpha \cdot e \cdot y_0$, $|\alpha| = 1$

$\operatorname{dist}(|c|y, Y_0) = \operatorname{dist}(\underbrace{\alpha|c|}_{c} y_0, \underbrace{y_0 Y}_{Y})$

$$\leq \operatorname{dist}(y, Y_0)^{-1} \|cy + y_0\|,$$ so $A = \operatorname{dist}(y, Y_0)^{-1}$ works

6) $\phi_0(y) = 1 \neq 0$

7) Use the Hahn-Banach theorem and extend ϕ_0 to the whole Y. ☐

We actually proved that the lemma holds $\forall y \in Y \setminus Y_0$.

We can now prove the theorem from above.

Theorem: Let X be a Banach space, $T \in \mathcal{B}(X)$. Then $\overline{TX} = \{x \in X \mid \langle x, \phi \rangle = 0 \ \forall \phi \in \operatorname{Ker} T\}$.

Proof: We have $TX \subset E$, $E = \{x \mid \langle x, \phi \rangle = 0 \ \forall \phi \in \operatorname{Ker} T^*\}$, because $\langle Tx, \phi \rangle = \langle x, T^*\phi \rangle = 0 \ \forall \phi \in \operatorname{Ker} T^*$.
Then $\overline{TX} \subset \overline{E} = E$ since E is closed.

$\left(\begin{matrix} x_n \to x \text{ and } \langle x_n, \phi \rangle = 0 \text{ for some } \phi \in X^* \\ \text{then } \langle x, \phi \rangle = \lim \langle x_n, \phi \rangle = 0 \end{matrix} \right)$

We now need to check $\overline{TX} \supset X$. If not, the inclusion $\overline{TX} \subset E$ is proper and by the seperation lemma $\exists \phi : \phi|_{\overline{TX}} \overset{(*)}{=} 0$ but $\phi(e) \neq 0$ for some $e \in E$.

$(*) \Rightarrow \phi|_{TX} = 0 \Leftrightarrow \langle Tx, \phi \rangle = 0 \ \forall x \in X \Leftrightarrow \langle x, T^*\phi \rangle = 0 \ \forall x \in X$
$\Leftrightarrow \phi \in \operatorname{Ker} T^* \Rightarrow \phi(e) = 0 \ \forall e \in E$ by definition of E, contradiction. ▨

Classical Form of Fredholm alternative for integral equations

Theorem: Let (S, μ) be a space with measure μ, and let $K(x,y) : S \times S \longrightarrow \mathbb{C}$:

$$\iint_{S \times S} |K(x,y)|^2 \, d\mu(x) \, d\mu(y) < \infty .$$

Then either equation $f(y) + \int_S K(x,y) f(x) \, d\mu(x) \overset{(**)}{=} 0$ has only the trivial solution $f = 0$ and the equation

$$f(y) + \int_S K(x,y) f(x) \, d\mu(x) = g(y)$$

is solvable for every $g \in L^2(S, \mu)$ or the equation $(**)$ has a non-trivial solution in $L^2(S, \mu)$.

Remark: Uniqueness implies existance.

Proof: Let's define $T = I - K$, $(Kf)(y) = \int_S K(x,y) f(x) \, d\mu(x)$.
Consider T as an operator on $L^2(S, \mu)$.

Then $(**) \Leftrightarrow \operatorname{Ker} T = \{0\} \Leftrightarrow TL^2(S,\mu) = L^2(S,\mu)$ by Fredholm alternative, modulo the fact that $K \in S_\infty(L^2(S,\mu))$. ▨

↑ This we pospone until Hilbert spaces theory.

Remarks: Further reading:

1) $\dim \operatorname{Ker} T = \dim \operatorname{Ker} T^*$ if $T = 1 - K$, $K \in S_\infty$, it coincides with the dimension of the space of solutions

2) There is a version of Fredholm theory for general operators $T \in \mathcal{B}(X)$: $\dim(\operatorname{Ker} T) < \infty$, $\dim(X/\operatorname{Ran} T) < \infty$.

Spectrum of compact operators

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Definition: Let $T \in \mathcal{B}(X)$, X Banach space.

$$\sigma(T) := \{\lambda \in \mathbb{C} \mid \lambda I - T \text{ is not invertible in } \mathcal{B}(X)\}$$

is called the spectrum of T.

Definition: A number $\lambda \in \mathbb{C}$ is called an eigenvalue of T if $\exists e \in X \setminus \{0\}.\ Te = \lambda e$.

Definition: $\sigma_p(T) := \{\lambda \text{ eigenvalue of } T\}$ ... point spectrum of T.

Remark: We have $\sigma_p(T) \subset \sigma(T)$ for every $T \in \mathcal{B}(X)$.

Proof: $\lambda \in \sigma_p(T) \Rightarrow \lambda I - T$ is not injective because $\operatorname{Ker}(\lambda I - T) \neq \{0\}$.

Remark: In general, we might have $\sigma_p(T) \neq \sigma(T)$ and even $\sigma_p(T) = \varnothing$.

Theorem: Let X be a Banach space, $\dim X = +\infty$, and let $K \in S_\infty(X)$. Then $0 \in \sigma(K)$, $\sigma(K) \setminus \{0\} \subset \sigma_p(K)$. Moreover, each eigenvalue has a finite multiplicity, and $\#\{\lambda \in \sigma_p(K) \mid |\lambda| \geq r\} < \infty$ for every $r > 0$.

Proof: $0 \in \sigma(K)$ since $0 \notin \sigma(K)$, then $\exists K^{-1} \in \mathcal{B}(X)$: $I = \underset{\overset{\Uparrow}{S_\infty(X)}}{K}\,\underset{\overset{\Uparrow}{\mathcal{B}(X)}}{K^{-1}} = K^{-1}K$, but then $I \in S_\infty(X) \Rightarrow \dim X < \infty$, contradiction.

Now let's prove that $\sigma(K) \setminus \{0\} \subset \sigma_p(K)$.
Take $\lambda \in \sigma(K)$ and assume that $\lambda \in \sigma_p(K)$. Then

$$\operatorname{Ker}(\lambda I - K) = \{0\} \Leftrightarrow \operatorname{Ker}(I - \tfrac{1}{\lambda}K) = \{0\}$$

$\Rightarrow I - \frac{1}{\lambda}K$ is invertible by Fredholm theorem – contradiction.

Now let's prove that $\dim E_\lambda < \infty$.

$$E_\lambda := \{e \in X \mid Ke = \lambda e\} \quad \left(\begin{array}{l}\text{definition for:}\\ \lambda \text{ has finite multiplicity}\end{array}\right)$$

If this is not the case, there is a sequence $\{e_n\}_{n=1}^{\infty}$ such that $\operatorname{dist}(e_n, \operatorname{span}\{e_1, \dots, e_{n-1}\}) \geq 1/2$, $\|e_n\| = 1$.
Consider $\{Ke_n\} = \{\lambda e_n\}$: we cannot choose a convergent subsequence from this sequence – contradiction with $K \in S_\infty(X)$.

To prove $\#\{\lambda \in \sigma_p(K) \mid |\lambda| \geq r\} < \infty$ for every $r > 0$, assume the converse and let $e_n \in X$: $Ke_n = \lambda_n e_n$, $\|e_n\| = 1$, $\lambda_n \in \sigma_p(K)$, $\lambda_n \neq \lambda_k$ for $k \neq n$.
Define $E_n := \operatorname{span}\{e_1, \dots, e_n\}$ $\forall n$.
Observation: $E_{n+1} \supsetneq E_n$ $\forall n$.
Clearly $E_{n+1} \supset E_n$. If $E_{n+1} = E_n$ for some n, there exists the first such n. Then

$$\lambda_{n+1} e_{n+1} = \sum_{k=1}^{n} \lambda_{n+1} \alpha_k e_k$$

$$\lambda_{n+1} e_{n+1} = \sum_{k=1}^{n} \alpha_k \lambda_k e_k$$

$$\Rightarrow 0 = \sum_{k=1}^{n} \alpha_k (\lambda_{n+1} - \lambda_k) e_k \Rightarrow \alpha_{n+1} - \lambda_n = 0$$

$$\Rightarrow \lambda_{n+1} = \lambda_n \text{ ... contradiction}$$

$\Rightarrow$ The observation is true.

Let's choose $y_{n+1} \in E_{n+1}$ such that $\|y_{n+1}\| = 1$, $\operatorname{dist}(y_{n+1}, E_n) \geq \frac{1}{2}$
It remains to prove that $\{Ky_n\}$ does not have a convergent subsequence.

$$y_{n+1} = \alpha_{n+1} e_{n+1} + R_n, \quad \text{where } R_n \in E_n$$

$$Ky_{n+1} - Ky_{m+1} = \lambda_{n+1} \alpha_{n+1} e_{n+1} + \tilde{R}_n - \lambda_{m+1} \alpha_{m+1} e_{m+1} - \tilde{R}_m$$

Assume that $n \geq m+1$, then

$$\underset{\substack{\cap \\ E_n}}{\tilde{R}_n} - \underbrace{\lambda_{m+1} \alpha_{m+1} e_{m+1}}_{\substack{\cap \\ E_{m+1} \subset E_n}} + \underset{\substack{\cap \\ E_m \subset E_n}}{\tilde{R}_m} \quad \in E_n$$

$$\begin{aligned} \Rightarrow \|Ky_{n+1} - Ky_{m+1}\| &\geq \operatorname{dist}(\lambda_{n+1} \alpha_{n+1} e_{n+1}, E_n) \\ &= |\lambda_{n+1}| \operatorname{dist}(\alpha_{n+1} e_{n+1}, E_n) \\ &= |\lambda_{n+1}| \operatorname{dist}(y_{n+1}, E_n) \\ &\geq r \cdot \tfrac{1}{2} > 0 \end{aligned}$$

▨

Scheme of the solution of Dirichlet problem in $\mathbb{R}^n$, $n \geq 3$, by means of Fredholm theory

Dirichlet problem: Find $u \in C^2(\Omega) \cap C(\bar{\Omega})$ $\left(\begin{matrix}\Omega \text{ is a domain in } \mathbb{R}^n \\ \partial\Omega \in C^2\end{matrix}\right)$ such that

$$\begin{cases} \Delta u = 0 \\ u|_{\partial\Omega} \overset{(*)}{=} f \end{cases} \quad \text{where } f \in C(\partial\Omega)$$

Physics interpretation: heat

Scheme for solution: we search for a solution of the form

$$u(x) = \int_{\partial\Omega} K(x,y)\varphi(y)\,dy, \quad K(x,y) = c_n \frac{(x-y,\ n_y)}{\|x-y\|^n_{\mathbb{R}^n}}$$

n_y is the ouward unit normal, $c_n \in \mathbb{R}$

$\Delta u = 0$ for every $\varphi \in C(\partial\Omega)$, we only need to find good φ (such that $(*)$ will hold)

$$u_\varphi(x) = -\varphi(x) + \int_{\partial\Omega} K(x,y)\varphi(y)\,dy \quad \text{if} \quad x \in \partial\Omega$$

$$u_\varphi = (-I + K)\varphi, \quad K\varphi = \int K(x,y)\varphi(y)\,dy$$

To check that $\exists \varphi$: $u_\varphi = f$ on $\partial\Omega$ we just check that $K \in S_\infty((C(\partial\Omega))$ and $\operatorname{Ker}(-I+K) = \{0\}$
$\Rightarrow$ we are done by Fredholm alternative.

This is hard to prove (course on PDEs)