# PART III. Applications

## Classical moment problems

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**Hamburger's moment problem:**
Given a sequence of numbers $\{h_k\}_{k=0}^{\infty}$, decide if there exists a measure $\mu$ on $\mathbb{R}$ s.t. $h_k = \int_{\mathbb{R}} x^k d\mu$, $k \geq 0$.
($\Leftrightarrow$ which sequences of reals are moment sequences of measures?)

**Trigonometric moment problem:**
Given a sequence $\{t_k\}_{k=-\infty}^{\infty} \subset \mathbb{C}$, is there a $\mu \geq 0$ s.t. $t_k = \int_{\mathbb{T}} z^k d\mu$?

Obvious restrictions:

- Hamburger case: $\sum_{k,j=0}^{N} h_{k+j} a_k \bar{a}_j \overset{(*)}{\geq} 0 \quad \forall \{a_k\}_{k=0}^{N} \subset \mathbb{C},\ N \geq 0$

$$0 \leq \int_{\mathbb{R}} \Big|\sum_{k=0}^{N} a_k x^k\Big|^2 d\mu = \sum_{1}^{N} a_k \bar{a}_j \int_{\mathbb{R}} x^{k+j} d\mu \overset{N}{=} \sum_{k,j=0} a_k \bar{a}_j h_{k+j}$$

- Trigonometric case: $\sum_{k,j=-N}^{N} t_{k-j} a_k \bar{a}_j \overset{(**)}{\geq} 0 \quad \forall \{a_k\}_{-N}^{N} \subset \mathbb{C}, N \geq 0$

$$0 \leq \int_{\mathbb{T}} \Big|\sum_{-N}^{N} a_k z^k\Big|^2 d\mu = \sum_{-N}^{N} a_k \bar{a}_j \int_{\mathbb{R}} z^k \bar{z}^j d\mu \overset{N}{=} \sum_{-N}^{N} a_k \bar{a}_j t_{k-j}$$

**Theorem [Hamburger]:** The assumption $(*)$ is sufficient for the solvability of the Hamburger case.

**Theorem:** The assumption $(**)$ is sufficient for the solvability of the Trigonometric moment problem. Moreover, we have
$\sum_{-N}^{N} t_{k-j} a_k \bar{a}_j > 0\ \forall \{a_k\}_{k=0}^{N} \Leftrightarrow \{t_k\}$ is the moment sequence of a measure $\mu$ such that $\#\operatorname{supp}\mu = +\infty$. $(***)$

Our goal is to prove $(***)$.

Proof: $H_0 = \left( \operatorname{span}\{z^k\}_{k\in\mathbb{Z}},\ \left\langle \sum_{-N}^{N} a_k z^k, \sum_{-N}^{N} \tilde{a}_k z^k \right\rangle := \sum t_{k-j}\, a_k \overline{\tilde{a}_j} \right)$

↳ pre Hilbert space, because it is linear, and $\langle \cdot,\cdot \rangle$ is the inner product on $H_0$, but $H_0$ is not complete w.r.t. $\|\sum a_z z^k\| = \sqrt{\left\langle \sum_{-N}^{N} a_z z^k, \sum_{-N}^{N} a_z z^k \right\rangle}$

General Functional analysis implies that $\exists H$ - Hilbert space such that $H_0 \subset H$ as a dense linear subset.

$T: \sum_{k=0}^{N} a_k z^k \longmapsto \sum_{k=0}^{N} a_k z^{k+1}$ - densely defined operator on $H$:

$$\left\| T\left(\sum_{-N}^{N} a_k z^k\right)\right\|^2 = \left\|\sum_{-N}^{N} a_k z^{k+1}\right\|^2 = \sum_{-N+1}^{N+1} t_{k-j}\, a_{k-1}\overline{a_{j-1}} = \sum_{-N+1}^{N+1} t_{(k-1)-(j-1)}\, a_{k-1}\overline{a_{j-1}} =$$

$$\underset{\sum_{-N+1}^{N+1} a_{k-1} z^k}{\|} \qquad = \sum_{-N}^{N} t_{k-j}\, a_k \overline{a_j} = \left\|\sum_{-N}^{N} a_k z^k\right\|^2$$

$\Rightarrow$ $T$ is an isometry initially defined on $H_0$.

Let's extend it to the whole space $H$. $\Rightarrow T$ is isometry on $H$, $T(H) \supset H_0$ - dense in $H$, since $T(H)$ is closed, we have $T(H) = H$.

$\Rightarrow T$ is unitary. Moreover, there is $h = 1$ s.t. $\underset{(H_0)}{\overset{\|}{\operatorname{span}\{T^k T^{*j} h\}}}$ is dense in $H$.

By the spectral theorem, there is a measure $\mu$ s.t. $\operatorname{supp}\mu = \mathbb{T} \subset \mathbb{T}$: $T \cong M_z$ on $L^2(\mu)$.

$$\langle T^k h, h\rangle_H = \langle M_z^k 1, 1\rangle_{L^2(\mu)} \ \forall k \geq 0 \text{ for } h = 1 \text{ in } H.$$

$$\|$$

$$\langle T^k 1, 1\rangle_H = \langle z^k, 1\rangle_H = \sum_{0}^{k} t_{i-j}\, \delta_k(i)\,\delta_0(j) = t_k$$

$$\langle M_z^k 1, 1\rangle_{L^2(\mu)} = \langle z^k, 1\rangle_{L^2(\mu)} = \int_{\mathbb{T}} z^k \, d\mu$$

$$\left.\begin{aligned} &\Rightarrow t_k = \int_{\mathbb{T}} z^k\, d\mu, \quad k \geq 0 \\ &\underline{t_{-k} = \overline{t_k}} = \overline{\int_{\mathbb{T}} z^k\, d\mu} = \int_{\mathbb{T}} z^{-k}\, d\mu, \ k \geq 0 \end{aligned}\right\} \Rightarrow t_k \text{ is the moment sequence}$$

$$\begin{cases} t_k = \langle z^k, 1\rangle \\ t_{-k} = \langle z^{-k}, 1\rangle = \langle T^k z^{-k}, T^k 1\rangle = \langle 1, z^k\rangle = \overline{\langle z^k, 1\rangle} = \overline{t_k} \Rightarrow \overline{t_{-k}} = t_k \end{cases}$$

It remains to show that the measure $\mu$ is such that $\#\operatorname{supp}\mu = \infty$.

$\Leftrightarrow \int \left|\sum_{-N}^{N} a_k z^k\right|^2 d\mu > 0$ (true by assumption). ∎

# Characters on compact Abelian groups

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Definition: $G$ is a topological group if $G$ is a group with topology whose operation is continuous in the product topology $G \times G$, and the operation of taking the inverses is also continuous.

Definition: $G$ is a compact group if $G$ is a topological group such that $G$ with its topology is a compact Hausdorff space.

Definition: A map $\gamma: G \longrightarrow \mathbb{T}$ is a character if $\gamma$ is a group homomorphism and $\gamma \in \mathcal{C}(G, \mathbb{T})$.

Remark: We will deal with the abelian (commutative) case, and we will denote the group operation by "+", the inverse element to $x \in G$ by $-x$, and the identity of the group by $0$.

Definition: $\hat{G} = \{\gamma \text{ character of } G\}$ is called the dual group to $G$.

Remark: In our notation, for every $\gamma \in \hat{G}$ we have

$$\left.\begin{aligned} &\gamma(x+y) = \gamma(x) \cdot \gamma(y) \quad \forall x,y \in G \\ &|\gamma(x)| = 1 \quad \forall x \in G \\ &\gamma \in \mathcal{C}(G, \mathbb{T}) \end{aligned}\right] \text{ equivalent to } \gamma \in \hat{G}$$

Remark: $\gamma_0 : x \longmapsto 1$ is always in $\hat{G}$

Definition: Let $G$ be a locally compact topological group. Then $\mu$ is the Haar measure on $G$ if $\mu(\mathcal{U}+x) = \mu(x+\mathcal{U}) = \mu(\mathcal{U})$ for every Borel set $\mathcal{U}$, $\mu \neq 0$, $\mu$ is regular ($\Rightarrow$ finite on compact subsets).

Theorem [Weyl]: Every locally compact topological group has a Haar measure $\mu$, which is unique up to multiplication by a constant.

**Agreement:** If $G$ is compact, we normalize $\mu$: $\mu(G)=1$. With this normalization the Haar measure is unique.

**Theorem [Peter-Weyl]:** If $G$ is a commutative compact group, then characters form an orthonormal basis in $L^2(G,\mu)$, where $\mu$ is the Haar measure of $G$.

Examples of characters:

- $G=\mathbb{R}$ (locally compact), $\hat{G}=\{e^{i\lambda x} \mid \lambda\in\mathbb{R}\}$, $\mathbb{R}\cong\hat{\mathbb{R}}$.
- $G=\mathbb{T}$ (compact), $\hat{\mathbb{T}}=\{z^n \mid n\in\mathbb{Z}\}$, $\hat{\mathbb{T}}\cong\mathbb{Z}$.
- $G=\mathbb{Z}/n\mathbb{Z}$ (compact), $\hat{G}=G_n$.
- $G_n=\{\xi\in\mathbb{T} \mid \xi^m=1\}$, $\hat{G}=\mathbb{Z}/n\mathbb{Z}$.

The decomposition of $f=\sum_{k\in\mathbb{Z}} c_k z^k$ for every $f\in L^2(\mathbb{T})$ is just the Fourier decomposition, the map $f\longmapsto\{c_k\}$ is the discrete Fourier transform. In the continuous case $(G=\mathbb{R})$ $f(x)=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} g(x)e^{ix\lambda}dx$ is the Fourier decomposition, and the map $f\longmapsto g$ is the Fourier transform $\left(g(t)=\int_{\mathbb{R}} f(x)e^{-ixt}dx\right)$.

Characters form an orthonormal system:

$$\int_G \gamma_1(x)\overline{\gamma_2(x)}\,d\mu \overset{\mu \text{ is Haar}}{=} \int_G \gamma_1(x-y)\overline{\gamma_2(x-y)}\,d\mu(x)$$
$$=\gamma_1(-y)\overline{\gamma_2(-y)}\int_G \gamma_1(x)\overline{\gamma_2(x)}\,d\mu$$

$$\Rightarrow \langle \gamma_1(x),\gamma_2(x)\rangle_{L^2(\mu)} = \gamma_1(-y)\overline{\gamma_2(-y)}\langle\gamma_1,\gamma_2\rangle_{L^2(\mu)}$$

$$\Rightarrow \langle\gamma_1(x),\gamma_2(x)\rangle\neq 0 \iff \gamma_1(-y)\overline{\gamma_2(-y)}=1 \quad \forall y\in G$$
$$\iff \gamma_1(-y)=\overline{\gamma_2(-y)} \quad \forall y\in G$$
$$\iff \gamma_1=\gamma_2$$

If $\gamma_1=\gamma_2=\gamma$, then $\|\gamma\|^2=\int_G \underset{=1}{|\gamma|^2}\,d\mu=\mu(G)=1$

So, the problem is completeness of $\{\gamma\}_{\gamma \in \hat{G}}$.

<u>Definition</u>: Let $s \in G$. The shift operator $T_s$ is $T_s : f \longmapsto f(\cdot - s)$.

<u>Lemma 1</u>: $T_s$ is unitary on $L^2(G) = L^2(G, \mu)$ ($\mu$ is Haar).

<u>Proof</u>: $T_s L^2(G) = L^2(G)$ and

$$\|T_s f\|^2_{L^2(G)} = \int_G |f(x-s)|^2 d\mu = \int_G |f(x)|^2 d\mu = \|f\|^2_{L^2(G)}$$

$\Rightarrow T_s$ is an isometry. $\blacksquare$

<u>Lemma 2</u>: For every $f \in L^2(G)$, we have $T_s f \longrightarrow f$ as $s \longrightarrow 0$ in $G$
($\Leftrightarrow \forall \varepsilon > 0 . \exists U_\varepsilon$ - open neighbourhood of $0 . \forall s \in U_\varepsilon . \|T_s f - f\| < \varepsilon$).

To prove this lemma, we will use a version of Cantors theorem for compact groups:

<u>Theorem</u>: If $f \in \mathcal{C}(G)$, $G$ is a compact group, then $\forall \varepsilon > 0 . \exists U_\varepsilon$ a neighbourhood of $0$ such that $|f(x) - f(y)| < \varepsilon$ if $x - y \in U_\varepsilon$. (Without proof.)

<u>Proof of lemma 2</u>: Take $f \in L^2(G)$ and find $g \in \mathcal{C}(G) : \|f - g\|_{L^2(G)} \leq \frac{\varepsilon}{3}$
(this is possible, because $\mu$ is a regular measure).

$$\|T_s f - f\|_{L^2(G)} \leq \underbrace{\|T_s(f-g)\|}_{\leq \varepsilon/3} + \underbrace{\|f-g\|}_{\leq \varepsilon/3} + \underbrace{\|T_s g - g\|}_{\leq ?}$$

$$\|T_s g - g\|_{L^2(G)} \leq \|T_s g - g\|_{L^\infty(G)} = \max_{x \in G} |g(x-s) - g(x)| \leq \varepsilon/3$$

by Cantors theorem if $s \in U_{\varepsilon/3}$ ($U_{\varepsilon/3}$ from Cantors theorem) $\blacksquare$

<u>Definition</u>: Let $f, g \in L^1(G)$, then $(f * g)(y) = \int_G f(x) g(y-x) d\mu(x)$.

<u>Remark</u>: $L^p(G) \subset L^1(G)$, because $G$ is compact, so

$$\left(\int |f| d\mu\right) \le \left(\int 1^q\right)^{1/q}\left(\int |f|^p\right)^{1/p} = \|f\|_{L^p(G)}$$

In particular, we can also define $f*g$ for every $f\in L^{p_1}(G)$, $g\in L^{p_2}(G)$

**Lemma 3 [Young inequality]:** $1\le p\le\infty$

$$\|f*g\|_{L^p} \le \|f\|_{L^p(G)}\|g\|_{L^1(G)} \quad \forall f\in L^p(G),\ g\in L^1(G).$$

**Proof:** We may assume that $\|g\|_{L^1(G)}=1$. Then

$$\|g*f\|^p_{L^p} = \int_G \Big|\int_G f(y-x)\underbrace{g(x)d\mu(x)}_{\text{probability measure}}\Big|^p d\mu(y) \overset{\text{Jensen}}{\le} \int_G\int_G |f(y-x)|^p |g(x)|\, d\mu(x)\, d\mu(y)$$

$$\overset{\text{Fubini}}{=} \int_G |g(x)| \underbrace{\int_G |f(y-x)|^p d\mu(y)}_{\int_G |f(y)|^p d\mu} d\mu(x) = \underbrace{\|g\|_{L^1}}_{=1}\cdot \|f\|^p_{L^p}$$

$\Rightarrow \|g*f\|_{L^p}\le \|f\|_{L^p} = \|g\|_{L^1}\cdot\|f\|_{L^p}$.

It remains to note that $g*f=f*g$:

$$f*g = \int f(x)\, g(y-x)\, d\mu(x) \underset{x=y-\tilde x}{=} \int f(y-\tilde x)\, g(\tilde x)\, d\mu(\tilde x)$$ ∎

**Lemma 4 [Approximation lemma]:** $\forall \varphi\in L^2(G,\mu)$ we have

$$\inf_{\substack{U\ni 0\\ U=-U\\ U \text{ open}}} \Big\|\varphi - \varphi * \frac{\chi_U}{\mu(U)}\Big\| = 0.$$

**Proof:** Take $\varphi\in L^2(G,\mu)$ and $\psi\in\mathcal{C}(G)$: $\|\varphi-\psi\|_{L^2(G)}<\varepsilon$.

$$\Big\|\varphi - \varphi*\frac{\chi_U}{\mu(U)}\Big\|_{L^2} \le \Big\|\psi-\psi*\frac{\chi_U}{\mu(U)}\Big\|_{L^2} + \underbrace{\|\varphi-\psi\|_2}_{\le\varepsilon} + \underbrace{\Big\|(\varphi-\psi)*\frac{\chi_U}{\mu(U)}\Big\|_{L^2}}_{\le \|\varphi-\psi\|_{L^2}\underbrace{\left\|\frac{\chi_U}{\mu(U)}\right\|_{L^1}}_{=1}\le\varepsilon}$$

$$\Big\|\psi-\psi*\frac{\chi_U}{\mu(U)}\Big\|_{L^2(G)} \le \Big\|\psi(y) - \int_G \psi(x)\frac{\chi_U(y-x)}{\mu(U)}\, d\mu\Big\|_{L^2(G)} \le$$

$$\le \sup_{y\in G}\int_G \underbrace{|\psi(y)-\psi(x)|}_{\le\varepsilon}\, \frac{\chi_U(y-x)}{\mu(x)}\, d\mu \le \varepsilon$$

if $U=U_\varepsilon$ for the function $f=\psi$ in Cantor's theorem ∎

**Goal:** $G$ - compact abelian group with Haar measure $\mu$, then $\hat{G}$ is an ONB in $L^2(G,\mu)$.

**Lemma:** Let $G$ be as above, $f \in L^2(G,\mu)$. Then F.A.E:
i) $f = c \cdot \gamma$, $\gamma \in \hat{G}$
ii) $T_s f = \lambda_s f$ in $L^2(G,\mu)$ $\forall s \in G$

$T_s f = f(\cdot - s)$, $s \in G$

**Proof:** $(1) \Rightarrow (2)$: $T_s(c\gamma) = c\gamma(x-s) = c\gamma(-s)\gamma(x)$, so $\lambda_s := \gamma(-s)$.
$(2) \Rightarrow (1)$: $T_{s+s'} = T_s T_{s'}$ $\quad s, s' \in G$
$\Rightarrow \lambda_{s+s'} f(x) = \lambda_s \lambda_{s'} f(x)$ for $\mu$-a.e. $x \in G$
$\Rightarrow \lambda_{s+s'} = \lambda_s \cdot \lambda_{s'}$ because $\exists x.\ f(x) \neq 0$ (otherwise $\underset{\gamma = 1}{}$ one can take $c=0$)

$\|T_s f - T_{s'} f\| = |\lambda_s - \lambda_{s'}|\,\|f\| \Rightarrow$ the map $s \mapsto \lambda_s$ is continuous from $G$ to $\mathbb{T}$:
$\Leftrightarrow |\lambda_s - \lambda_{s'}| \longrightarrow 0$ if $s' \longrightarrow s$ in $G$ $\Leftrightarrow \|T_s f - T_{s'} f\|_{L^2(\mu,G)} \longrightarrow 0$ if $s' \longrightarrow s$,
which is true for $f \in \mathcal{C}(G)$ by Cantor's theorem and
$$\|T_s f - T_{s'} f\| \le \|T_s(f - \tilde{f})\| + \|T_{s'}(f - \tilde{f})\| + \|T_s \tilde{f} - T_{s'} \tilde{f}\|$$
$$\le \underbrace{2\|f - \tilde{f}\|}_{\le \frac{\varepsilon}{4}\ \forall s'} + \underbrace{\|T_s \tilde{f} - T_{s'} \tilde{f}\|}_{\le \frac{\varepsilon}{2} \text{ for } s' \text{ close to } s}$$

$\Rightarrow \lambda_s$ is a continuous function from $G$ to $\mathbb{C}$

$$\left.\|f\| \underset{\substack{\uparrow \\ \text{isometricity} \\ \text{of } T_s}}{=} \|T_s f\| \underset{\substack{\uparrow \\ \text{by } T_s f = \lambda_s f}}{=} |\lambda_s| \cdot \|f\| \right\} \Rightarrow |\lambda_s| = 1 \text{ for every } s \in G$$

$\Rightarrow \lambda_s \in \mathcal{C}(G, \mathbb{T})$, $\lambda_{s+s'} = \lambda_s \cdot \lambda_{s'} \Rightarrow \lambda_s \in \hat{G}$.
Let us prove that $f = c\,\overline{\lambda_x}$ for some $c \in \mathbb{C}$ $\quad (\gamma := \overline{\lambda_x})$.
$h(x) := \lambda_x f(x)$. We have $T_s h = \lambda_{x-s} f(x-s) = \lambda_x \lambda_s f(x) = \lambda_x f(x) = h$ on a set $E_s \subset G$: $\mu(E_s) = \mu(G) = 1$. Unfortunately $E_s$ might depend on $s$ and we cannot say $h(x) = h(0 - (-x)) \underset{\uparrow \text{problem}}{=} (T_{-x} h)(0) = h(0)$. So we need to argue differently.

Take $g = \frac{\chi_U}{\mu(U)}$ for some $U$ - open set in $G$, $U = -U$:
$$(h * g)(y) = \int_G h(x) g(y-x)\, d\mu(x) \underset{\substack{\uparrow \\ \mu \text{ is Haar}}}{=} \int_G h(x+y) g(y - (x+y))\, d\mu(x)$$

$$= \int_G h(x)g(-x)\,d\mu(x) \overset{?}{\underset{u:=-u}{=}} \int h(x)g(x)\,d\mu = C_u$$

$\Rightarrow (h*g)(y) = C_u \ \forall y \in G$. In fact, $C_u = C$, $C$ does not depend on $u$:

$$C_u = \int_G C_u\,d\mu = \int_G\int_G h(x)g(y-x)\,d\mu\,d\mu \overset{\text{Fubini}}{=} \int h(x)\Big(\underbrace{\int_G g(y-x)\,d\mu(y)}_{\int_G g(y)\,d\mu(y) = 1}\Big)\,d\mu(x)$$

$$= \int_G h(x)\,d\mu = C$$

$\Rightarrow h * \frac{\chi_u}{\mu(u)} = C$, $C$ does not depend on $u$.

From the approximation lemma, $\inf_h \|h - h*\frac{\chi_u}{\mu(u)}\|_{L^1(\mu)} = 0$ we have $\|h-C\|_{L^1(\mu)} = 0 \Rightarrow h = C$ a.e. on $G$.

$\Rightarrow \lambda_x \varphi_x = C$ a.e. on $G$

$\Rightarrow \varphi = C\cdot\gamma$ for $\gamma = \overline{\lambda_x}$. ▨

<u>Lemma</u>: $A: f \longmapsto f * \frac{\chi_u}{\mu(u)}$, $\overset{u=-u}{\mu(u)>0}$ – a compact self-adjoint operator on $L^2(G,\mu)$.

<u>Proof</u>: $Af = \int_G k(x,y)f(x)\,d\mu(x)$ for $k(x,y) = \frac{\chi_u(y-x)}{\mu(u)}$

Note that $k(x,y) = k(y,x) = \overline{k(y,x)} \Rightarrow A = A^*$ $\left(A^*f = \int \overline{k(y,x)}f(x)\,d\mu(x)\right)$.

$$\iint_{G\times G} |k(x,y)|^2\,d\mu(x)\,d\mu(y) < \infty \quad \left(\text{in our case, } \iint_{G\times G}|k(x,y)|^2\,d\mu\,d\mu = \frac{1}{\mu(u)} < \infty\right)$$

$\Rightarrow$ From Hilbert-Schmidt test (to be proved later) $A \in S_\infty(L^2(G,\mu))$. ▨

<u>Lemma</u>: If $H$ is a separable Hilbert space, $A \in S_\infty(H)$, $A = A^* \Rightarrow$ $A = \sum_{\lambda_k \in \sigma(A)} \lambda_k P_{E_k}$, where $E = \{h \in H \mid Ah = \lambda_k h\}$, and the series converges in operator norm.

<u>Proof</u>: This is an exercise from the homework.

<u>Lemma</u>: Let $H$ be a finite-dimensional Hilbert space, $\dim H = N < \infty$. Let $\{U_\alpha\}_{\alpha\in I}$ be a family of unitary operators on $H$, $U_\alpha U_\beta = U_\beta U_\alpha$ $\forall \alpha,\beta \in I$. $\Rightarrow \exists \{e_n\}_{n=1}^N$ – an ONB in $H$: $U_\alpha e_n = \lambda_{\alpha,n} e_n \ \forall \alpha \in I$.

Proof: Induction on $N$.

- $N=1$ $\quad U_\alpha = c_\alpha \cdot I \quad \forall \alpha$ ✓
- $N-1 \longrightarrow N$, $N \geq 2$

either $U_\alpha = c_\alpha I \quad \forall \alpha$ ✓

or $\exists \alpha.\ E_\alpha = \{h \in H \mid U_\alpha h = \lambda_\alpha h\}$ satisfies $E_\alpha \neq \{0\}$, $E_\alpha \neq H$

$\forall \beta \in I$, $\forall h \in E_\alpha$, we have $U_\alpha(U_\beta h)) = U_\beta(U_\alpha h) = U_\beta(\lambda_\alpha h) = \lambda_\alpha U_\beta h$

$\Rightarrow U_\beta h \in E_\alpha$ by definition of $E_\alpha$

$\Rightarrow U_\beta E_\alpha \subset E_\alpha \Rightarrow U_\beta E_\alpha = E_\alpha \quad (\dim U_\beta E_\alpha = \dim E_\alpha)$.

Moreover, $U^*_\beta E_\alpha = U^{-1}_\beta(E_\alpha) = E_\alpha$. So, $E_\alpha$ is a reducing subspace for $U_\beta$, in particular, $U_\beta = (U_\beta|_{\underbrace{E_\alpha}_{\dim \leq N-1}}) \oplus (U_B|_{\underbrace{E_\alpha^\perp}_{\dim \leq N-1}}) \quad \forall \beta \in I$.

$\Rightarrow$ by induction assumption, ok. ∎

Proof of Peter-Weyl: We need to prove that $\hat{G}$ is an ONB in $L^2(G,\mu)$.
We know that $\forall \gamma \neq \tilde{\gamma} \quad (\gamma, \tilde{\gamma})_{L^2(G,\mu)} = 0$, so we need to check that $\forall \varphi \in L^2(G,\mu).\ \varphi \in \operatorname{clos}_{L^2(\mu)}(\operatorname{span} \hat{G})$.

Take an open neighbourhood $U$ of $0$ s.t. $U = -U$. Consider the compact self-adjoint operator $A_U : f \longmapsto f * \frac{\chi_U}{\mu(U)}$. We have $A_U = \sum_{\lambda_k \in \sigma(A_U)} \lambda_k P_{E_{\lambda_k}}$

Let us show that $A_U \varphi \in \operatorname{clos}_{L^2(\mu)}(\operatorname{span} \hat{G})$. It is enough to check that $P_{E_{\lambda_k}} \varphi \in \operatorname{span} \hat{G}$. Observe that $T_s A_U = A_U T_s$:

$$A_U T_s f = \int f(x-s) \frac{\chi_U(y-x)}{\mu(U)} = \int f(x-s) \frac{\chi_U(y-s-(x-s))}{\mu(U)} d\mu = \int_G f(\tilde{x}) \frac{\chi_U(y-s-\tilde{x})}{\mu(U)} d\mu = T_s A_U f$$

$\Rightarrow$ If $h : A_U h = \lambda_k h \Rightarrow A_U T_s h = T_s A_U h = T_s(\lambda_k h) = \lambda_k T_s h \Rightarrow T_s h \in E_k$

$\Rightarrow T_s E_{\lambda_k} \subset E_{\lambda_k} \Rightarrow [\dim E_{\lambda_k} < \infty$ because $A$ is compact$] \Rightarrow T_s E_{\lambda_k} = E_{\lambda_k}$.

Now we can use lemma for $\{U_\alpha\}_{\alpha \in I} = \{T_s|_{E_{\lambda_k}}\}_{s \in G}$. There is a ONB $\{e_{\lambda_k,n}\} : T_s e_{\lambda_k,n} = h_s e_{\lambda_k,n} \quad \forall n \leq \dim E_{\lambda_k} \quad \forall s \in G$.

By lemma, $e_{\lambda_k,n} = c_{\lambda_k,n} \gamma_{\lambda_k,n}$ for some $c_{\lambda_k,n} \in \mathbb{C}$, $\gamma_{\lambda_k,n} \in \hat{G}$.

In particular $E_{\lambda_k}$ is spanned by $\{\gamma_{\lambda_k,n}\}_{n \leq \dim E_{\lambda_k,n}} \Rightarrow P_{E_{\lambda_k}} \varphi \in E \subset \operatorname{span} G$.

So, $A_U \varphi \in \operatorname{clos}_{L^2(G,\mu)}(\operatorname{span} \hat{G})$. By the approximation lemma,
$\inf_U \|A_U \varphi - \varphi\|_{L^2(G)} = 0$ and $A_U \varphi \in \operatorname{clos} \operatorname{span} \hat{G} \Rightarrow \varphi \in \operatorname{clos} \operatorname{span} \hat{G}$. ∎

# Minimax principle

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**Theorem [minimax principle]:** Let $H$ be a separable Hilbert space, $A \in S_\infty(H)$, $A = A^*$, $\sigma(A) = \{-\lambda_n^-\}_{n=1}^{N_-} \cup \{\lambda_n^+\}_{n=1}^{N_+}$ where $N_\pm \in \mathbb{N} \cup \{\infty\} \cup \{0\}$, $\lambda_n^\pm > 0$ $\forall n$, and each point $\lambda \in \sigma(A)$ appears in $\{-\lambda_n^-\} \cup \{\lambda_n^+\}$ exactly $k(\lambda)$ times, where $k(\lambda)$ is the multiplicity of $\lambda = \dim\{\varphi \mid A\varphi = \lambda\varphi\}$. Assume, moreover, that $\{\lambda_n^+\}$, $\{\lambda_n^-\}$ are non-increasing. Then

$$\pm\lambda_n^\pm = \min_{\substack{L \subset H \\ \operatorname{codim} L \le n-1}} \max_{x \in L \setminus \{0\}} \frac{\pm\langle Ax, x\rangle}{\langle x, x\rangle}. \qquad (\operatorname{codim} L = \dim(H \ominus L))$$

Proof: Let's consider the decomposition $A = \bigoplus_{\lambda \in \sigma(A)} \lambda P_{E_\lambda}$ ($E_\lambda = \{\varphi \mid A\varphi = \lambda\varphi\}$, see exercises).

Let's choose orthonormal sequence $\varphi_1^\pm, \varphi_2^\pm, \dots$ such that

$$A = -\sum_{n=1}^{N_-} \lambda_n^- \langle \cdot, \varphi_n^-\rangle \varphi_n^- + \sum_{n=1}^{N_+} \lambda_n^+ \langle \cdot, \varphi_n^+\rangle \varphi_n^+. \qquad (*)$$

Here we use the fact that in $E \subset H$ - subspace, $\{\psi_1, \dots, \psi_m\}$ - ONB in $E$, then $P_E = \sum_1^m \langle \cdot, \psi_k\rangle \psi_k$ - orthogonal projector in $H$ to $E$ (Proof: add orthonormal sequence $\psi_1^\perp, \psi_2^\perp, \dots$ so that $\underbrace{\{\psi_k\} \cup \{\psi_k^\perp\}}_{\{e_k\}}$ is ONB in $H$ and consider the action $P_E$ on $h = \sum_{k=1}^\infty c_k e_k$

Let's prove that $\lambda_n^+ = \min_{\substack{L \subset H \\ \operatorname{codim} L \le n-1}} \max_{x \in L \setminus \{0\}} \frac{\langle Ax, x\rangle}{\langle x, x\rangle}$.

(the case $-\lambda_n^- = \dots$ follows from this, because $-\lambda_n^-(A) = \lambda_n^+(-A)$, see $(*)$)

Set $F_n = \operatorname{span}\{\varphi_k^+\}_{k=1}^n$. For every $L \subset H$ : $\operatorname{codim} L \le n-1$ we have $F_n \cap L \ne \{0\}$. $(**)$

$$H = L \oplus \underbrace{(H \ominus L)}^{\dim \le n-1}$$
$$H = (H \ominus F_n) \oplus \underbrace{F_n}_{\dim n}$$

Indeed $P_{H\ominus L} : F_n \longrightarrow H \ominus L$ has a nonzero kernel $\Rightarrow \exists h \in F_n \setminus \{0\}. P_{H\ominus L} h = 0$ $\Leftrightarrow h \in L \Rightarrow h \in F_n \cap L$, $(**)$ ok.

Take $h \in (F_n \cap L) \setminus \{0\}$ and consider $h = \sum_{k=1}^n d_k \varphi_k^+$, $d_k \in \mathbb{C}$ $(h \in F_n)$.

$$\langle Ah, h\rangle \overset{(***)}{=} \left\langle \sum_{k=1}^{n} \lambda_k^+ \langle h, e_k^+\rangle, h\right\rangle$$

$$= \left\langle \sum_{k=1}^{n} \lambda_k^+ \alpha_k e_k^+, \sum_{k=1}^{n} \alpha_k e_k^+ \right\rangle$$

$$= \sum_{k=1}^{n} \lambda_k^+ |\alpha_k|^2$$

$$\geq \lambda_n^+ \Big( \underbrace{\sum_{k=1}^{n} |\alpha_k|^2}_{\hookrightarrow \text{norm}} \Big) \qquad \text{(sequence is non-increasing)}$$

$$= \lambda_n^+ \langle h, h\rangle$$

$$\Rightarrow \min_{\substack{L \subset H \\ \operatorname{codim} L \leq n-1}} \max_{h \neq 0} \frac{\langle Ah, h\rangle}{\langle h, h\rangle} \geq \lambda_n^+$$

To prove the converse inequality we take $L := H \ominus F_{n-1}$. For every $h \in H \ominus F_{n-1}$ we have

(here we use the fact that $\{e_k^+\}_1^{N_+} \cup \{e_k^-\}_1^{N_-}$ is an ONB in $H \ominus \ker A$, exercise)

$$P_{(\ker A)^\perp} h = \sum_{k=n}^{N_+} \alpha_k e_k^+ + \sum_{k=1}^{N_-} \beta_k e_k^-.$$

$$\langle Ah, h\rangle = \langle A\tilde{h}, \tilde{h}\rangle = \text{...argument similar to } (***)\text{...}$$

$$= \sum_{k=n}^{N_+} \lambda_k^+ |\alpha_k^2| - \underbrace{\sum_{k=1}^{N_-} \lambda_k^- |\beta_k|^2}_{\leq 0}$$

$$\leq \lambda_n^+ \cdot \sum_{k=n}^{N_+} |\alpha_k^2|$$

$$= \lambda_n^+ \langle h, h\rangle, \quad \text{equality holds for } h = e_n^+$$

For this $L = H \ominus F_{n-1}$ we proved $\max_{h \neq 0} \frac{\langle Ah, h\rangle}{\langle h, h\rangle} \leq \lambda_n^+$, ok. ▨

# Square root of a nonnegative operator

<u>Definition</u>: $A \in \mathcal{B}(H)$ is nonnegative if $\langle Ax, x\rangle \geq 0 \ \forall x \in H$

positive if $\langle Ax, x\rangle > 0 \ \forall x \in H$

<u>Theorem</u>: For every $A \in \mathcal{B}(H)$ s.t. $A \geq 0$ $\exists!$ $\sqrt{A}$ s.t.

1) $\sqrt{A} \in \mathcal{B}(H)$.

2) $\sqrt{A} \geq 0$.

3) $\sqrt{A}\sqrt{A} = A$.

Lemma 1 [polarization identity]: For every $A \in \mathcal{B}(H)$, $\forall x, y \in H$ we have

$$\langle Ax, y\rangle = \left\langle \frac{A(x+y)}{2}, \frac{x+y}{2}\right\rangle - \left\langle A\frac{x-y}{2}, \frac{x-y}{2}\right\rangle + i\left(\left\langle A\frac{x+iy}{2}, \frac{x+iy}{2}\right\rangle - \left\langle A\frac{x-iy}{2}, \frac{x-iy}{2}\right\rangle\right).$$

Proof: $\forall z \in \mathbb{C}$ we have $z = \operatorname{Re} z + i \operatorname{Im} z = \operatorname{Re} z + i \operatorname{Re}(-iz)$

$$\begin{aligned}\langle Ax, y\rangle &= \operatorname{Re}\langle Ax, y\rangle + i\operatorname{Re}(-i\langle Ax, y\rangle)\\ &= \operatorname{Re}\langle Ax, y\rangle + i\operatorname{Re}(\langle Ax, iy\rangle)\end{aligned}$$

$$\operatorname{Re}\langle Ax, y\rangle = \frac{1}{4}\left(\langle A(x+y), x+y\rangle - \langle A(x-y), x-y\rangle\right)$$ ▨

Corollary 1: Let $A_1, A_2 \in \mathcal{B}(H)$ : $\langle A_1 x, x\rangle = \langle A_2 x, x\rangle\ \forall x \in H \Rightarrow A_1 = A_2$.

Proof: By Lemma 1, $\langle A_1 x, y\rangle = \langle A_2 x, y\rangle\ \forall x, y \in H$,

$$\|A_1 - A_2\| = \sup_{\substack{\|x\|\le 1\\ \|y\|\le 1}} |\langle (A_1 - A_2)x, y\rangle| = 0 \Rightarrow A_1 = A_2$$ ▨

Corollary 2: $A \geq 0 \Rightarrow A = A^*$.

Proof: $\langle Ax, x\rangle \underset{\uparrow}{=} \langle x, Ax\rangle = \langle A^* x, x\rangle\ \forall x \in H \Rightarrow A = A^*$ by Corollary 1. ▨

$\langle Ax, x\rangle = \overline{\langle Ax, x\rangle}$

Proof of theorem: Take $A \geq 0$, by Corollary 2 we have $A = A^*$.
By spectral theorem, $A \cong \bigoplus_k M_{x_k}$, where $M_{x_k}$ is the multiplication operator $f \longmapsto xf$ on $L^2(\mu_k)$ for some $\mu_k \geq 0$, $\operatorname{supp}\mu_k \subset \mathbb{R}$.

$$\begin{aligned}
&A \geq 0 \Rightarrow M_{x_k} \geq 0\\
&\Leftrightarrow \langle xf, f\rangle_{L^2(\mu_k)} \geq 0 \quad \forall f \in L^2(\mu_k)\\
&\Leftrightarrow \int_{\mathbb{R}} x|f(x)|^2 \, d\mu_k \geq 0 \quad \forall f \in L^2(\mu_k)\\
&\Leftrightarrow \operatorname{supp}\mu_k \subset [0, +\infty)\\
&\Rightarrow \sigma(A) = \overline{\bigcup \operatorname{supp}\mu_k} \subset [0, \infty)
\end{aligned}$$

Now consider $f = \sqrt{x} \in C(\sigma(A))$. $\sqrt{A} := f(A)$

Then $\sqrt{A} \in \mathcal{B}(H)$ because $\|\sqrt{A}\| = \|f\|_{C(\sigma(A))}$

$$\sqrt{A}\cdot\sqrt{A} = f(A)\cdot f(A) = f^2(A) = x(A) = A$$

$$\sqrt{A} \geq 0 \text{ because } f(A) = \bigoplus_k M_{\sqrt{x_k}} \text{ and } \langle \sqrt{x} f, f\rangle_{L^2(\mu)} \geq 0 \quad \forall f \in L^2(\mu_k)$$

Uniqueness: Suppose there is $\widetilde{\sqrt{A}}$ with the same properties 1)-3).
$K := \sigma(\sqrt{A}) \cup \sigma(\widetilde{\sqrt{A}})$ - compact in $\mathbb{R}_+ = [0,\infty)$.
Find $p_n$ - polynomials: $p_n(x) \rightrightarrows \sqrt{x}$ on $[0, L] \subseteq K$, $L \geq 1$.
Define $q_n(x) := p_n(x^2)$, then $q_n(x) \rightrightarrows x$ on $[0, L^2] \supset K$. We have

$$\|q_n(\sqrt{A}) - \sqrt{A}\| \longrightarrow 0$$
$$\|q_n(\widetilde{\sqrt{A}}) - \widetilde{\sqrt{A}}\| \longrightarrow 0$$

by functional calculus

But $q_n(\sqrt{A}) = p_n((\sqrt{A})^2) = p_n(A) = p_n((\widetilde{\sqrt{A}})^2) = q_n(\widetilde{\sqrt{A}})$ $\forall n$
$\Rightarrow \sqrt{A} = \widetilde{\sqrt{A}}$. ▨

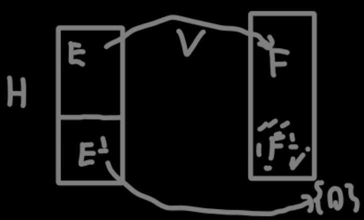
# Polar decomposition of bounded operators

January 7, 2026

Motivation: $z = |z|\cdot e^{i\operatorname{Arg} z}$, $z \in \mathbb{C}$

$\overset{?}{} \; T = V|T|$, $T \in \mathcal{B}(H)$

Definition: Let $H$ be a Hilbert space and $E, F \subset H$ subspaces in $H$. $V \in \mathcal{B}(H)$ is a partial isometry with domain of isometricity $E$ and range $F$, if $V|_E$ is a unitary operator from $E$ onto $F$. and $V|_{E^\perp} = 0$.




Theorem [polar decomposition]: $\forall T \in \mathcal{B}(H)$ $\exists$ partial isometry $V$ with domain of isometricity $\overline{\operatorname{Ran} T^*}$ and the range $\overline{\operatorname{Ran} T}$ such that $T = V|T|$, $|T| = \sqrt{T^*T}$. Moreover $\overline{\operatorname{Ran} T^*} = \overline{\operatorname{Ran}|T|}$.

**Remark:** $T^*T \geq 0$, so $|T| = \sqrt{T^*T}$ is defined correctly
$$\langle T^*Th, h\rangle = \langle Th, Th\rangle = \|Th\|^2 \geq 0$$

**Remark:** We might have $|T^*| \neq |T|$.

Proof of theorem: For every $T \in \mathcal{B}(H)$ we have
$$H = \overline{\operatorname{Ran} T^*} \oplus \operatorname{Ker} T \quad \left(\begin{array}{l} h \perp \overline{\operatorname{Ran} T^*} \Leftrightarrow h \perp \operatorname{Ran} T^* \Leftrightarrow \langle h, T^*H\rangle = 0 \\ \quad \Leftrightarrow \langle Th, H\rangle = 0 \Leftrightarrow Th = 0 \Leftrightarrow h \in \operatorname{Ker} T \end{array}\right)$$
From this formula, $\overline{\operatorname{Ran}|T|} = \overline{\operatorname{Ran} T^*}$. Indeed this is equivalent to $\operatorname{Ker}(|T|) \overset{(*)}{=} \operatorname{Ker} T$, but
$$\underset{0}{\underset{\parallel}{\langle |T|x, |T|x\rangle}} = \langle |T|^2 x, x\rangle \underset{\text{def. of } |T|}{=} \langle T^*Tx, x\rangle = \underset{0}{\underset{\parallel}{\langle Tx, Tx\rangle}},$$
so $(*)$ is ok.
Now define $V$ on $\operatorname{Ran}|T|$ as follows:
$$V: |T|x \longrightarrow Tx, \quad x \in H.$$
1) The definition is correct: if
$$\begin{array}{ccc} |T|x_1 = |T|x_2 & \Rightarrow & Tx_1 = Tx_2 \\ \Updownarrow & & \Updownarrow \\ |T|(x_1 - x_2) = 0 & & T(x_1 - x_2) = 0 \\ \Updownarrow & & \Updownarrow \\ x_1 - x_2 \in \operatorname{Ker}|T| & \overset{\text{by } (*)}{\Longleftrightarrow} & x_1 - x_2 \in \operatorname{Ker} T \end{array}$$

2) $V$ is linear - ok.
3) $V$ is an isometry on $\operatorname{Ran}|T|$:
$$\|V(|T|x)\|^2 = \langle Tx, Tx\rangle = \langle T^*Tx, x\rangle = \langle |T|^2 x, x\rangle = \langle |T|x, |T|x\rangle = \||T|x\|^2$$
Next, extend $V$ to $\overline{\operatorname{Ran}|T|} = \overline{\operatorname{Ran} T^*}$ by continuity and set $V|_{\overline{\operatorname{Ran} T^*}^{\perp}} = 0$. Then $V$ is a partial isometry with the domain of isometricity $\overline{\operatorname{Ran} T^*}$ and range $V = \overline{\operatorname{Ran} T}$. We also have $V|T|x = Tx$ by the definition of $V$. ▨

**Remark:** If $\dim \operatorname{Ker} T = \dim \operatorname{Ker} T^*$ then $\exists\, \mathcal{U}$ - unitary operator on $H$ s.t. $T = \mathcal{U}|T|$. (exercise)

# Singular numbers of compact operators. Finite rank approximation.

**Problem I:** Let $T \in \mathcal{B}(H)$. What is $\operatorname{dist}(T, \mathcal{F}_n)$ where $\mathcal{F}_n = \{A \in \mathcal{B}(H) \mid \operatorname{rank} A \leq n\}$.

**Problem II:** How to choose best approximant to $T$ from $\mathcal{F}_n$?

**Definition:** Let $T \in S_\infty(H)$, $|T| = \sqrt{T^*T}$, and let $\{\lambda_k(|T|)\}_{k=1}^{N_+}$, $N_+ \in \mathbb{Z}_+ \cup \{\infty\}$ be the sequence of eigenvalues, enumerated so that $\lambda_n \geq \lambda_{n+1}$ $\forall n$, and so that each eigenvalue $\lambda \in \sigma(|T|)$ appears in $\{\lambda_k(|T|)\}_{k=1}^{N_+}$ exactly $k(\lambda)$ times, where $k(\lambda)$ is the multiplicity of $\lambda$ $= \dim\{\varphi \mid |T|\varphi = \lambda\varphi\}$.
Then $s_k(T) = \lambda_k(|T|)$ is called the k-th singular value of $T$.

**Remark:** $s_k(T) = \sqrt{\lambda_k(T^*T)}$ because $f(\sigma(A)) = \sigma(f(A))$ $\forall A \in S_\infty(H)$, $f \in C(\sigma(A))$

(take $f = \sqrt{x}$, $A = T^*T = |T|^2 \geq 0$)

(spectral mapping theorem)

January 13, 2026

**Theorem:** Let $T \in S_\infty(H)$. Then $\exists \{\varphi_k\}$ ONB in $\overline{\operatorname{Ran} T^*}$, $\{\psi_k\}$ ONB in $\overline{\operatorname{Ran} T}$ such that $T = \sum s_k(T)\langle \cdot, \varphi_k\rangle \psi_k$, where the series converges in the operator norm. Conversely if $\{s_k\} \subset \mathbb{C}$ such that $s_k \downarrow 0$ (or $|\{s_k\}| < \infty$), then $T = \sum s_k \langle \cdot, \varphi_k\rangle \psi_k$ is compact for every pair of orthonormal sequences of the same cardinality.
Such a decomposition is called the Schmidt decomposition of a compact operator $T$.

**Proof:** Let us consider the polar decomposition $T = V|T|$. Since $|T| = |T^*|$ and $|T| = \sqrt{T^*T} = \lim p_n(T^*T)$ for $p_n$

polynomials such that $p_n \rightrightarrows \sqrt{x}$ on $\sigma(|T|)$, we have $|T| \in S_\infty(H)$.

$$\Rightarrow |T| = \sum_{\lambda \in \sigma(|T|)} \lambda P_{E_\lambda} = \sum \lambda_k(|T|) \langle \cdot, \varphi_k \rangle \varphi_k = \sum s_k(T) \langle \cdot, \varphi_k \rangle \varphi_k,$$

where $\{\varphi_k\}$ is an ONB in $\overline{\operatorname{Ran}|T|} = (\operatorname{Ker}|T|)^\perp$, and we know from the proof of polar decomposition that $\overline{\operatorname{Ran}|T|} = \overline{\operatorname{Ran}T^*}$, $V: \overline{\operatorname{Ran}T^*} \longrightarrow \operatorname{Ran}T$ is unitary, so $\psi_k := V\varphi_k \Rightarrow |\{\psi_k\}| = |\{\varphi_k\}|$ and $\{\psi_k\}$ is also an ONB in $\overline{\operatorname{Ran}T}$.

$$\Rightarrow T = V|T| = V\left(\sum s_k(T) \langle \cdot, \varphi_k \rangle \varphi_k\right) = \sum s_k(T) \langle \cdot, \varphi_k \rangle V\varphi_k$$
$$= \sum s_k(T) \langle \cdot, \varphi_k \rangle \psi_k$$

and the series converges in operator norm because $\sum s_k(T) \langle \cdot, \varphi_k \rangle \varphi_k$ converges in operator norm.

Conversely, let $T = \sum_{k=1}^{\infty} s_k \langle \cdot, \varphi_k \rangle \psi_k$ (for finite sums there is nothing to prove).

$\Rightarrow$ Set $T_n := \sum_{k=1}^{n} s_k \langle \cdot, \varphi_k \rangle \psi_k$. Then

$$\|T_n x - T_{n+\tilde{k}} x\|^2 = \left\| \sum_{n+1}^{n+\tilde{k}} s_k \langle x, \varphi_k \rangle \psi_k \right\|^2$$
$$= \sum_{n+1}^{n+\tilde{k}} s_k^2 |\langle x, \varphi_k \rangle|^2$$
$$\leq s_{n+1}^2 \sum_{n+1}^{n+\tilde{k}} |\langle x, \varphi_k \rangle|^2$$
$$\leq s_{n+1}^2 \|x\|^2$$

$\Rightarrow \|T_n - T_{n+\tilde{k}}\| \leq s_{n+1} \longrightarrow 0 \Rightarrow$ so the sequence is Cauchy

$\Leftrightarrow T = \sum_1^\infty s_k \langle \cdot, \varphi_k \rangle \psi_k$ converges in $\mathcal{B}(H)$. ▨

**Theorem:** Let $T \in S_\infty(H)$. Then $s_{n+1}(T) = \operatorname{dist}(T, \mathcal{F}_n)$, where $\mathcal{F}_n = \{K \mid \operatorname{rank} K \leq n\}$. Moreover, $\operatorname{dist}(T, \mathcal{F}_n) = \|T - T_n\|$, where $T_n = \sum_1^n s_k(T) \langle \cdot, \varphi_k \rangle \psi_k$.

**Lemma** [minimax principle for singular numbers]:

$$s_{n+1}(T) = \min_{\operatorname{codim} L \leq n} \max_{x \in L\setminus\{0\}} \frac{\|Tx\|}{\|x\|}.$$

**Proof:** $s_{n+1} = \lambda_{n+1}(|T|) = \lambda_{n+1}(\sqrt{TT^*}) \overset{\text{spectral theorem } f(\sigma(A)) = \sigma(f(A))}{=} \sqrt{\lambda_{n+1}(TT^*)}$

$$= \sqrt{\min_{\operatorname{codim} L \leq n} \max_{x \in L\setminus\{0\}} \frac{\langle T^*Tx, x\rangle}{\langle x, x\rangle}}$$

$$= \sqrt{\min\max \frac{\|Tx\|^2}{\|x\|^2}}$$

▨

**Proof of theorem:** Take $K \in \mathcal{F}_n$.

$\Rightarrow s_{n+1}(T) \leq \underbrace{\max_{x \in L\setminus\{0\}} \frac{\|(T-K)x\|}{\|x\|}}_{\|T-K\|}$ for $L = \ker K$
(codim $L \leq n$ because $K \in \mathcal{F}_n$)

$\Rightarrow s_{n+1}(T) \overset{(**)}{\leq} \|T-K\| \quad \forall K \in \mathcal{F}_n$

But $s_{n+1}(T) \underset{(*)}{\geq} \|T - T_n\|$, because $\forall x \in H.\ \|(T-T_n)x\|^2 = \|\sum_{n+1}^{\infty} s_k(T)\langle x, \varphi_k\rangle \psi_k\|^2$

$$= \sum_{n+1}^{\infty} s_k(T)^2 |\langle x, \varphi_k\rangle|^2$$

$$\leq s_{n+1}(T)^2 \|x\|^2$$

$\Rightarrow \|T - T_k\|^2 \leq s_{n+1}^2(T)$

$\Rightarrow \operatorname{dist}(T, \mathcal{F}_n) \leq \|T - T_n\| \leq s_{n+1}(T) \overset{(**)}{\leq} \operatorname{dist}(T, \mathcal{F}_n)$

$\Rightarrow s_{n+1} = \|T - T_n\| \overset{(*)}{=} \operatorname{dist}(T, \mathcal{F}_n)$. The theorem follows.

▨

**Definition:** Let $1 \leq p < \infty$. $S_p(H) := \{T\text{-compact on } H \mid \sum s_k(T)^p < \infty\}$ is the Schatten-Von Neumann class.

This is a Banach space wrt. the norm $\|T\|_{S_p} = \left(\sum s_k(T)^p\right)^{1/p}$.

**Remark:** $S_\infty$ is a "limit point case" of the scale $S_p$

$$S_1(H) \subseteq S_p(H) \subseteq S_\infty(H) \quad \forall\, 1 \le p \le \infty$$

**Proposition:** $S_p(H)$ is indeed a Banach space.

**Proof:**
$$\begin{aligned} s_{n+1}(\alpha T_1 + \beta T_2) &= \mathrm{dist}(\alpha T_1 + \beta T_2, \mathcal{F}_n) \\ &\le |\alpha|\,\mathrm{dist}(T_1, \mathcal{F}_n) + |\beta|\,\mathrm{dist}(T_2, \mathcal{F}_n) \\ &= |\alpha|\, s_{n+1}(T_1) + |\beta|\, s_{n+1}(T_2) \end{aligned}$$

So, if $T_1, T_2 \in S_p$, $\alpha, \beta \in \mathbb{C}$ $\Rightarrow \alpha T_1 + \beta T_2 \in S_p$ and

$$\|\alpha T_1 + \beta T_2\|_{S_p(H)} \le |\alpha| \cdot \|T_1\|_{S_p(H)} + |\beta| \cdot \|T_2\|_{S_p(H)}$$

$\Rightarrow \|\cdot\|_{S_p(H)}$ is a norm and $S_p(H)$ is linear.

Completness of $S_p(H) \Leftrightarrow \sum_1^\infty T_n \in S_p(H)$ $\forall T_n \in S_p(H)$, $\sum \|T_n\|_{S_p(H)} < \infty$.

But this holds because

$$\left.\begin{aligned} \|\textstyle\sum T_n\|_{S_p} &= \|\{\underbrace{s_k(\textstyle\sum_1^\infty T_n)}\}_{\ell^p}\| < \infty \\ &\hookrightarrow \le \textstyle\sum s_k(T_n) \end{aligned}\right\} \text{ because } \ell^p \text{ is complete}$$

▨

**Theorem:** For every $1 \le p \le \infty$, $S_p(H)$ is a two-sided symmetric ideal in $\mathcal{B}(H)$:

(1) $R, L \in \mathcal{B}(H)$, $T \in S_p(H) \Rightarrow LTR \in S_p(H)$, $\|LTR\|_{S_p(H)} \le \|L\| \cdot \|T\|_{S_p(H)} \cdot \|R\|$.

(2) $T \in S_p(H) \Leftrightarrow T^* \in S_p(H)$.

**Proof:** For $p = \infty$ we already know this, so let $1 \le p < \infty$, and consider $L, R \in \mathcal{B}(H)$.

$$\begin{aligned} s_{n+1}(LTR) &= \inf_{K \in \mathcal{F}_n} \|LTR - K\| \le \inf \|LTR - LKR\| \\ &\le \|L\|\,\mathrm{dist}(T, \mathcal{F}_n) \cdot \|R\| = \|L\| \cdot s_{n+1}(T) \cdot \|R\| \end{aligned}$$

$\Rightarrow$ (1) ✓

(2) $s_{n+1}(T) = \mathrm{dist}(T, \mathcal{F}_n) = \mathrm{dist}(T^*, \mathcal{F}_n^*) = \mathrm{dist}(T^*, \mathcal{F}_n) = s_{n+1}(T^*)$ ▨

**Definition:** $S_1(H)$ - trace class, $S_2(H)$ - Hilbert-Schmidt class.

**Definition:** $T_1 \in S(H)$, $\operatorname{tr} T := \sum_{k=1}^{\infty} \langle Te_k, e_k\rangle$ where $\{e_k\}$ is an ONB in H.
trace

**Lemma:** The value of $\operatorname{tr} T$ does not depend on $\{e_k\}$.

**Proof:** Take $\{e_k\}$ and note that $|\operatorname{tr} T| \leq \|T\|_{S_1}$.

$$\sum_{k=1}^{\infty}\Big(\sum_j s_j \langle e_k, \varphi_j\rangle\langle \psi_j, e_k\rangle\Big) \leq \sum_j s_j \sum_k |\langle e_k, \varphi_j\rangle|\cdot|\langle \psi_j, e_k\rangle|$$
$$\leq \sum_j s_j \Big(\sum_k |\langle e_k, \varphi_j\rangle|^2\Big)^{1/2}\Big(\sum_k |\langle \psi_j, e_k\rangle|^2\Big)^{1/2}$$
$$\leq \sum_j s_j = \|T\|_{S_1}$$

Next, $\operatorname{tr} T = \operatorname{tr}(T - T_n) + \operatorname{tr} T_n$, $T_n = \sum_1^n s_k \langle\cdot, \varphi_k\rangle \psi_k$.
So it remains to prove that $\sum_{k=1}^{\infty}\langle T_n e_k, e_k\rangle$ does not depend on $\{e_k\}$. This follows from the fact that $\sum_{k=1}^{\infty}\langle Se_k, e_k\rangle$ does not depend on $\{e_k\}$.
But $\sum_{k=1}^{\infty}\langle Se_k, e_k\rangle = \sum_{k=1}^{\infty}\langle e_k, \varphi\rangle\langle \psi, e_k\rangle$
$$= \Big\langle \sum_1^{\infty}\langle \psi, e_k\rangle e_k, \sum_1^{\infty}\langle \varphi, e_k\rangle e_k\Big\rangle$$
$= \langle \psi, \varphi\rangle$ does not depend on $\{e_k\}$. ▨

January 14, 2025

**Theorem:** If $A, B \in S_1(H)$, then $\operatorname{trace}(AB) = \operatorname{trace}(BA)$.

**Proof:** $A = \sum_{k=1}^{\infty} s_k \langle\cdot, \varphi_k\rangle \psi_k$

$$\operatorname{trace}(AB) = \sum \langle AB\psi_{\tilde{k}}, \psi_{\tilde{k}}\rangle = \sum_{\tilde{k}=1}^{\infty}\Big\langle \sum_{k=1}^{\infty} s_k \langle B\psi_{\tilde{k}}, \varphi_k\rangle \psi_k, \psi_{\tilde{k}}\Big\rangle$$

add some vectors $\psi_k$ to the initial basis in the Schmidt decomposition so that the resulting sequence will be an ONB, set $s_k = 0$ for the new vectors $\psi_k$

$$= \sum_{\tilde{k}=1}^{\infty} s_{\tilde{k}} \langle B\psi_{\tilde{k}}, \varphi_{\tilde{k}}\rangle$$

$$\operatorname{trace}(BA) \underset{\substack{\text{add same}\\ \text{as above}}}{=} \sum \langle BA\varphi_{\tilde{k}}, \varphi_{\tilde{k}}\rangle = \sum_{\tilde{k}} \langle B \underbrace{\Big(\sum_k s_k \langle \varphi_{\tilde{k}}, \varphi_k\rangle \psi_k\Big)}_{s_{\tilde{k}}\psi_{\tilde{k}}}, \varphi_{\tilde{k}}\rangle$$

$$= \sum_{\tilde{k}} \langle B s_{\tilde{k}} \psi_{\tilde{k}}, \varphi_{\tilde{k}}\rangle$$

$\Rightarrow \operatorname{trace}(AB) = \operatorname{trace}(BA)$. ▨

**Theorem [Lidski]**: $\operatorname{trace}(A) = \sum \lambda_k(A)$, $\{\lambda_k(A)\}$ is the set of eigenvalues counted with multiplicities, $\operatorname{trace} A = 0$ if $\nexists$ eigenvalues.

Important result - without proof.

**Theorem**: $T \in S_2(H) \Leftrightarrow \sum_{k=1}^{\infty} \|Te_k\|^2 < \infty$ for some $\{e_k\}$-ONB in $H$. Moreover, $\sum_{k=1}^{\infty} \|Te_k\|^2 = \|T\|^2_{S_2(H)} = \sum s_k^2(T)$. In particular, it does not depend on $\{e_k\}$.

**Proof**: $T \in S_\infty(H)$, $T = \sum s_k \langle \cdot, \varphi_k\rangle \psi_k$

$$\sum_j \|Te_j\|^2 = \sum_j \Big\| \sum_k s_k \langle e_j, \varphi_k\rangle \psi_k \Big\|^2 = \sum_j \sum_k s_k^2 |\langle e_j, \varphi_k\rangle|^2$$

$$= \sum_k s_k^2 \|\varphi_k\|^2 = \sum_{k=1}^{\infty} s_k^2 = \|T\|^2_{S_2(H)}$$ ▨

**Proposition**: $T_1, T_2 \in S_2(H) \Rightarrow T_1T_2 \in S_1(H)$.

**Proof**: Take $\{u_k\}, \{v_k\}$-ONB in $H$.

$$\sum |\langle T_1T_2u_k, v_k\rangle| = \sum |\langle T_2u_k, T_1^* v_k\rangle| \leq \sum \|T_2u_k\| \cdot \|T_1^* v_k\|$$

$$\leq \sqrt{\sum \|T_2u_k\|^2 \cdot \sum \|T_1^* v_k\|^2} = \|T_2\|_{S_2} \cdot \|T_1^*\|_{S_2} = \|T_1\|_{S_1} \cdot \|T_2\|_{S_2}$$

So, we checked that $\sum |\langle T_1T_2u_k, v_k\rangle| \leq \|T_1\| \cdot \|T_2\|$

$T_1T_2 = \sum s_k(T_1T_2) \langle \cdot, \varphi_k, \psi_k)$

$$\sum s_k(T_1T_2) = \sum s_k(T_1T_2) \langle \varphi_k, \varphi_k\rangle \langle \psi_k, \psi_k\rangle$$

$$= \sum (T_1T_2\varphi_k, \psi_k) < \infty$$

$\Rightarrow \sum s_k(T_1T_2) < \infty \Rightarrow T_1T_2 \in S_1$ ▨

**Remark:** $T_1 \in S_p$, $T_2 \in S_q$, $\frac{1}{p}+\frac{1}{q}=1 \Rightarrow T_1T_2 \in S_1$ (exercise).

**Theorem:** $S_2(H)$ is the Hilbert space with the inner product $\langle A, B\rangle = \operatorname{trace}(AB^*)$.

**Proof:** $\langle \alpha A_1 + \beta A_2, B\rangle = \alpha\langle A_1, B\rangle + \beta\langle A_2, B\rangle$

$\langle A, B\rangle = \overline{\langle B, A\rangle}$

$\langle A, A\rangle = \operatorname{trace}(AA^*) = \sum \langle AA^* e_k, e_k\rangle = \sum \|A^* e_k\|^2 = \|A^*\|_{S_2}^2 = \|A\|_{S_2}^2$ ▨

**Theorem:** Let $\{u_k\}_1^\infty$ be an ONB in $H$. Then $\{\langle \cdot, u_k\rangle u_j\}_{1\le k,j<\infty}$ is an ONB in $S_2(H)$.

**Proof:** $\operatorname{span}\{\langle \cdot, u_k\rangle u_j\} = \bigcup_{n=0}^{\infty} \mathcal{F}_n$ ... dense in $S_2(H)$

$(T_n \to T$ in $S_2$ $\forall T \in S_2(H))$

↑ piece of Schmidt decomposition

$$\langle \underbrace{\langle \cdot, u_k\rangle u_j}_{T_{kj}}, \underbrace{\langle \cdot, u_m\rangle u_s}_{T_{ms}}\rangle = \operatorname{trace}(T_{kj} T_{ms}^*) =: \gamma(k,j,m,s)$$

Take $h$, $T_{ms}^* h = \langle h, u_s\rangle u_m$

$$T_{kj}(T_{ms}^* h) = \langle\langle h, u_s\rangle u_m, e_k\rangle u_j = \langle h, u_s\rangle \underbrace{\langle u_m, u_k\rangle}_{c_{mk}} u_j$$
$$= (c_{mk}\langle \cdot, u_s\rangle u_j)(h)$$

$$\operatorname{trace}(c_{mk}\langle \cdot, u_s\rangle u_j) = \sum_\ell \langle c_{mk}\langle u_\ell, u_s\rangle u_j, u_\ell\rangle$$
$$= c_{mk}\langle u_j, u_s\rangle \underbrace{\langle u_j, u_j\rangle}_{1} = c_{mk}\langle u_j, u_s\rangle$$
$$= \langle u_m, u_k\rangle\langle u_j, u_s\rangle = \begin{cases} 1; & k=m,\ j=s \\ 0; & \text{otherwise} \end{cases} = \gamma(k,j,m,s)$$ ▨

**Theorem:** $(S_p(H))^* = S_q(H)$, $\frac{1}{p}+\frac{1}{q}=1$, $1<p<\infty$

$(S_1(H))^* = \mathcal{B}(H)$

$(S_\infty(H))^* = S_1(H)$

The pairing is given by $\langle A, B\rangle = \operatorname{tr}(AB^*)$.

**Theorem [Hilbert-Schmidt kernels]:** Assume that $\mu$ is a Borel measure on $X \subset \mathbb{R}^n$ such that $\# \operatorname{supp} \mu = \infty$. Then $S_2(L^2(\mu)) \cong L^2(\mu \times \mu)$ i.e., the operator

$$\mathcal{U} : \sum c_{kj} T_{kj} \longmapsto \sum u_j(x) \overline{u_k(y)}$$

is the iso from $S_2(L^2(\mu))$ onto $L^2(\mu \times \mu)$.

In particular, $T \in S_2(L^2(\mu)) \Leftrightarrow \exists K(x,y) \in L^2(\mu \times \mu)$;

$$Tf = \int K(x,y) f(y) \, d\mu(y),$$

moreover $\|T\|_{S_2}^2 = \iint |K(x,y)|^2 \, d\mu(x) \, d\mu(y)$.